Preliminary Examination - Numerical Analysis - August, 2019

Instructions: Solve 6 out of 8 problems. Any results that your responses rely upon must be stated clearly.

1. Let $s_\Delta : [-1,1] \to \mathbb{R}$ be a cubic spline function relative to the domain partition $\Delta \equiv \{-1,0,1\}$, so there is one *knot* at $x = 0$. Consider the set V of all such s_Δ that satisfy the simultaneous conditions $s_\Delta(-1) = 0$, $s_\Delta(1) = 0$, $s_\Delta{}'(-1) = 0$ and $s_\Delta{}'(1) = 0$. Note that V is a vector space. Derive a basis for V (show the work for your derivation).

2. Given any $f : [a,b] \to \mathbb{R}$ that is continuous on $[a,b]$, $-\infty < a < b < \infty$, denote the integral of f by

$$I(f) = \int_a^b f(x)\,dx$$

and let the two-point Gaussian quadrature approximation (Gauss-Legendre, specifically) be denoted by

$$I_2(f) = \sum_{i=1}^{2} w_i f(x_i),$$

with weights $w_i > 0$ and abscissa $x_i \in (a,b)$, for $i = 1,2$. In the case

$$f(x) = \left(x - \frac{a+b}{2}\right)^4,$$

derive the explicit, algebraic formula for the quadrature error $I(f) - I_2(f)$ in terms of a and b.

3. Given any integer $n \geq 0$, let $\mathbb{P}_n[-1,1]$ denote the space of all real-valued polynomials of degree n or less on the domain $-1 \leq x \leq 1$. Let the subset of polynomials with lead coefficient 1 be denoted by $\tilde{\mathbb{P}}_n[-1,1]$:

$$\tilde{\mathbb{P}}_n[-1,1] = \left\{p : [-1,1] \to \mathbb{R} \mid p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0,\ \{c_{n-1},\ldots,c_0\} \subset \mathbb{R}\right\}.$$

Given any $p \in \tilde{\mathbb{P}}_{n+1}[-1,1]$, let $q \in \mathbb{P}_n[-1,1]$ denote the best *uniform* approximant of p in $\mathbb{P}_n[-1,1]$. Prove that $p - q$ is given by a multiple of a certain Chebyshev polynomial.

4. Let N be a positive integer and $\Delta x = 2\pi/N$ be a uniform spacing for points $x_j = -\pi + j\Delta x$, $j = 0,1,\ldots,N$. Given $f : [-\pi,\pi] \to \mathbb{R}$, continuous, let us denote by $I(f)$ the integral

$$I(f) = \int_{-\pi}^{\pi} f(x)\,dx.$$

Relative to the given domain partition, denote the composite trapezoidal approximation of $I(f)$ by $\tilde{I}(f)$. Prove that for $f(x) = \sin(x)$, the quadrature rule exhibits "superconvergence", in the sense that

$$\lim_{\Delta x \to 0} \frac{I(f) - \tilde{I}(f)}{\Delta x^2} = 0.$$

5. In this problem, $i = \sqrt{-1}$. Let $N > 1$ be an integer and define points $x_n = 2\pi n/N$, for $n = 0,1,\ldots,N-1$. Consider a function

$$\phi_m(x) = \frac{\sin(x)}{N} \sum_{k=0}^{N-1} e^{ik(x-x_m)},$$

for some fixed integer m, $0 \le m < N$. There is a unique *phase polynomial* of the form

$$p(x) = \sum_{j=0}^{N-1} \beta_j e^{ijx}$$

with the property that $p(x_n) = \phi_m(x_n)$ for all $n = 0, 1, \dots, N-1$. Provide explicit formulas for the coefficients β_j, $0 \le j \le N-1$, in terms of the subindex values j and m. No other indices should appear in your formulas (note that i and N are not indices here).

6. Denote by $\|\cdot\|_V$ some vector norm on $\mathbb{C}^n$, and let M_n be the space of all complex, square matrices with n rows. Given any $\mathbf{A} \in M_n$, denote the induced matrix norm by

$$\text{lub}_V(\mathbf{A}) = \max_{\|\vec{x}\|_V=1} \|\mathbf{A}\vec{x}\|_V.$$

(a) Prove that $\text{lub}_V(\cdot)$ is submultiplicative.

(b) Given any matrix norm $\|\cdot\|$ on M_n that is consistent with $\|\cdot\|_V$, prove that the condition number, $\kappa(\mathbf{A})$, defined with respect to the matrix norm $\|\cdot\|$ for an invertible $\mathbf{A}$ satisfies

$$1 \le \kappa(\mathbf{A}).$$

(c) Now let the vector norm $\|\vec{x}\|_V$ be the maximum size of the entries of $\vec{x}$. In the linear system $\mathbf{A}\vec{x} = \vec{b}$, with $\mathbf{A}$ defined below, assume that the vector $\vec{b}$ may have up to a 10% relative error, as measured using the vector norm $\|\cdot\|_V$. Provide a numerical bound (with justification) for the corresponding relative error in computing the entries of $\vec{x}$.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

7. Let $w(x) = |x|$ be a weight function on the domain $-1 \le x \le 1$. Some w-orthogonal polynomials on the indicated domain can be defined as

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= x, \\ \phi_2(x) &= x\phi_1(x) - \frac{1}{2}\phi_0(x), \\ \phi_3(x) &= x\phi_2(x) - \frac{1}{6}\phi_1(x). \end{aligned}$$

Given a continuous function $f : [-1, 1] \to \mathbb{R}$, let $I(f)$ be the integral

$$I(f) = \int_{-1}^{1} w(x)\, f(x)\, dx.$$

Denote by $\tilde{I}(f)$ the the 3-point Guassian quadrature rule to approximate $I(f)$. Specifically,

$$\tilde{I}(f) = \sum_{i=1}^{3} w_i f(x_i) \approx I(f),$$

where $\tilde{I}(f) = I(f)$ whenever f is a polynomial of order 5 or less. Derive the numerical values of w_i and x_i for $i = 1, 2, 3$.

8. Let $f(x) = x^4$. Denote by $p(x)$ the cubic polynomial such that $p(-1) = f(-1)$, $p(1) = f(1)$,

$p'(1) = f'(1)$ and $p''(1) = f''(1)$. Let $q(x)$ be the cubic polynomial that interpolates $f(x)$ at $x = -1, 0, 1, 2$.
(a) Provide the explicit polynomial $p(x)$.

(b) Provide the explicit polynomial $q(x)$.

(c) Prove the following:

$$\lim_{x \to -1} \frac{f(x) - p(x)}{f(x) - q(x)} = \frac{4}{3}.$$