Instructions: Solve 6 out of 8 problems. Any results that your responses rely upon must be stated clearly.

1. Let \( s_\Delta : [-1, 1] \to \mathbb{R} \) be a cubic spline function relative to the domain partition \( \Delta \equiv \{-1, 0, 1\} \), so there is one knot at \( x = 0 \). Consider the set \( V \) of all such \( s_\Delta \) that satisfy the simultaneous conditions \( s_\Delta(-1) = 0 \), \( s_\Delta(1) = 0 \), \( s_\Delta'(-1) = 0 \) and \( s_\Delta'(1) = 0 \). Note that \( V \) is a vector space. Derive a basis for \( V \) (show the work for your derivation).

2. Given any \( f : [a, b] \to \mathbb{R} \) that is continuous on \(-\infty < a < b < \infty\), denote the integral of \( f \) by \( I(f) = \int_a^b f(x) \, dx \) and let the two-point Gaussian quadrature approximation (Gauss-Legendre, specifically) be denoted by \( I_2(f) = \sum_{i=1}^{2} w_i f(x_i) \), with weights \( w_i > 0 \) and abscissa \( x_i \in (a, b) \), for \( i = 1, 2 \). In the case \( f(x) = (x - \frac{a + b}{2})^4 \), derive the explicit, algebraic formula for the quadrature error \( I(f) - I_2(f) \) in terms of \( a \) and \( b \).

3. Given any integer \( n \geq 0 \), let \( \mathbb{P}_n[-1, 1] \) denote the space of all real-valued polynomials of degree \( n \) or less on the domain \(-1 \leq x \leq 1\). Let the subset of polynomials with lead coefficient 1 be denoted by \( \tilde{\mathbb{P}}_n[-1, 1] \): 
\[
\tilde{\mathbb{P}}_n[-1, 1] = \{ p : [-1, 1] \to \mathbb{R} \mid p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0, \{c_{n-1}, \ldots, c_0\} \subset \mathbb{R} \}.
\]
Given any \( p \in \tilde{\mathbb{P}}_{n+1}[-1, 1] \), let \( q \in \mathbb{P}_n[-1, 1] \) denote the best uniform approximant of \( p \) in \( \mathbb{P}_n[-1, 1] \). Prove that \( p - q \) is given by a multiple of a certain Chebyshev polynomial.

4. Let \( N \) be a positive integer and \( \Delta x = 2\pi/N \) be a uniform spacing for points \( x_j = -\pi + j\Delta x \), \( j = 0, 1, \ldots, N \). Given \( f : [-\pi, \pi] \to \mathbb{R} \), continuous, let us denote by \( I(f) \) the integral
\[
I(f) = \int_{-\pi}^{\pi} f(x) \, dx.
\]
Relative to the given domain partition, denote the composite trapezoidal approximation of \( I(f) \) by \( \tilde{I}(f) \). Prove that for \( f(x) = \sin(x) \), the quadrature rule exhibits “superconvergence”, in the sense that
\[
\lim_{\Delta x \to 0} \frac{I(f) - \tilde{I}(f)}{(\Delta x)^2} = 0.
\]

5. In this problem, \( i = \sqrt{-1} \). Let \( N > 1 \) be an integer and define points \( x_n = 2\pi n/N \), for \( n = 0, 1, \ldots, N - 1 \). Consider a function
\[
\phi_m(x) = \frac{\sin(x)}{N} \sum_{k=0}^{N-1} e^{ik(x-x_m)},
\]
for some fixed integer \( m, 0 \leq m < N \). There is a unique phase polynomial of the form

\[
p(x) = \sum_{j=0}^{N-1} \beta_j e^{ijx}
\]

with the property that \( p(x_n) = \phi_m(x_n) \) for all \( n = 0, 1, \ldots, N - 1 \). Provide explicit formulas for the coefficients \( \beta_j, 0 \leq j \leq N - 1 \), in terms of the subindex values \( j \) and \( m \). No other indices should appear in your formulas (note that \( i \) and \( N \) are not indices here).

6. Denote by \( \| \cdot \|_V \) some vector norm on \( \mathbb{C}^n \), and let \( M_n \) be the space of all complex, square matrices with \( n \) rows. Given any \( \mathbf{A} \in M_n \), denote the induced matrix norm by

\[
\text{lub}_V (\mathbf{A}) = \max_{\|\mathbf{x}\|_V = 1} \| \mathbf{A}\mathbf{x}\|_V.
\]

(a) Prove that \( \text{lub}_V (\cdot) \) is submultiplicative.

(b) Given any matrix norm \( \| \cdot \|_\cdot \) on \( M_n \) that is consistent with \( \| \cdot \|_V \), prove that the condition number, \( \kappa(\mathbf{A}) \), defined with respect to the matrix norm \( \| \cdot \|_\cdot \) for an invertible \( \mathbf{A} \) satisfies

\[
1 \leq \kappa(\mathbf{A}).
\]

(c) Now let the vector norm \( \| \mathbf{x}\|_V \) be the maximum size of the entries of \( \mathbf{x} \). In the linear system \( \mathbf{A}\mathbf{x} = \mathbf{b} \), with \( \mathbf{A} \) defined below, assume that the vector \( \mathbf{b} \) may have up to a 10% relative error, as measured using the vector norm \( \| \cdot \|_V \). Provide a numerical bound (with justification) for the corresponding relative error in computing the entries of \( \mathbf{x} \).

\[
\mathbf{A} = \begin{bmatrix}
2 & -1 \\
0 & 1
\end{bmatrix}.
\]

7. Let \( w(x) = |x| \) be a weight function on the domain \(-1 \leq x \leq 1\). Some \( w \)-orthogonal polynomials on the indicated domain can be defined as

\[
\begin{align*}
\phi_0(x) &= 1, \\
\phi_1(x) &= x, \\
\phi_2(x) &= x\phi_1(x) - \frac{1}{2}\phi_0(x), \\
\phi_3(x) &= x\phi_2(x) - \frac{1}{6}\phi_1(x).
\end{align*}
\]

Given a continuous function \( f : [-1, 1] \to \mathbb{R} \), let \( I(f) \) be the integral

\[
I(f) = \int_{-1}^{1} w(x) f(x) \, dx.
\]

Denote by \( \tilde{I}(f) \) the the 3-point Guassian quadrature rule to approximate \( I(f) \). Specifically,

\[
\tilde{I}(f) = \sum_{i=1}^{3} w_i f(x_i) \approx I(f),
\]

where \( \tilde{I}(f) = I(f) \) whenever \( f \) is a polynomial of order 5 or less. Derive the numerical values of \( w_i \) and \( x_i \) for \( i = 1, 2, 3 \).

8. Let \( f(x) = x^4 \). Denote by \( p(x) \) the cubic polynomial such that \( p(-1) = f(-1) \), \( p(1) = f(1) \),
\( p'(1) = f'(1) \) and \( p''(1) = f''(1) \). Let \( q(x) \) be the cubic polynomial that interpolates \( f(x) \) at \( x = -1, 0, 1, 2 \).

(a) Provide the explicit polynomial \( p(x) \).

(b) Provide the explicit polynomial \( q(x) \).

(c) Prove the following:

\[
\lim_{x \to -1} \frac{f(x) - p(x)}{f(x) - q(x)} = \frac{4}{3}.
\]