Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on \( \mathbb{R}^n \) is denoted by \( m \) or \( dx \). The \( \sigma \)-algebra of Borel sets in \( \mathbb{R}^n \) is denoted by \( \mathcal{B}(\mathbb{R}^n) \). The characteristic function of a set \( A \) is denoted by \( \chi_A \). If \( S \subset \mathbb{R}^n \) is Lebesgue measurable then \( L^p(S) := L^p(m|_S) \) where \( m|_S \) is the restriction of the Lebesgue measure \( m \) on \( S \).

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1. (10 points) Let \( 0 < a < b < \infty \), and consider the function

\[
f(x) = \frac{1}{x^a + x^b}, \quad x > 0.
\]

For what values of \( p \) does \( f \in L^p((0,\infty)) \)?

2. (10 points) Let \( g : [0, 1] \to \mathbb{R} \) be a nonnegative Lebesgue measurable function.

(a) Prove that, as \( n \to \infty \), the numbers \( I_n = \int_{[0,1]} g^n \, dm \) converge to a non-negative limit that may be infinite.

(b) If \( I_n = C < \infty \) for all \( n \in \mathbb{N} \), show that there exists some Lebesgue measurable set \( A \subset [0, 1] \) such that \( g = \chi_A \), \( m \)-a.e.

3. (10 points) Let \( \mu, \nu \) be two measures on a measurable space \((X, \mathcal{A})\). Prove that if for every \( \epsilon > 0 \) there exists a measurable set \( A_\epsilon \) such that \( \mu(A_\epsilon) < \epsilon \) and \( \nu(A_\epsilon) < \epsilon \) then \( \mu \perp \nu \).

4. (15 points) Prove or disprove three of the following statements.

(a) If \( (f_n)_{n \in \mathbb{N}}, f_n : \mathbb{R} \to \mathbb{R} \), is a sequence of measurable functions such that \( f_n \to 0 \) in \( L^1(\mathbb{R}) \) and in \( L^2(\mathbb{R}) \) then \( f_n \to 0 \) in \( L^4(\mathbb{R}) \).

(b) There exists a function \( f : [0, 1] \to \mathbb{R} \) such that \( f(0) = 0, f(1) = 1 \) and

\[
\sup_{x \neq y, |x - y| < 1} \frac{|f(x) - f(y)|}{|x - y|} < 1.
\]

(c) There exists a measurable set \( A \subset [0, 1] \) with \( m(A) = 0.9 \) such that \( m(A \cap I) > 0.1 \, m(I) \) for every open interval \( I \subset [0, 1] \).

(d) There exists a probability measure \( \mu \) on \( \mathcal{B}(\mathbb{R}^n) \) such that \( \mu([x]) = 0 \) for all \( x \in \mathbb{R}^n \) and \( \mu(B) \in [0, 1] \) for \( B \in \mathcal{B}(\mathbb{R}^n) \).

5. (10 points) Let \( A \subset \mathbb{R}^n \) be a Lebesgue measurable set such that \( m(A) < \infty \) and let \( t \in (0, m(A)/2) \). Prove that there exist disjoint Lebesgue measurable sets \( B, C \subset A \) such that \( m(B) = m(C) = t \).

6. (10 points) Prove that \( L^\infty([0,1]) \) is a set of first category in the space \( (L^1([0,1]), \| \cdot \|_1) \).

*Hint:* Consider the sets \( E_n = \{ f \in L^\infty([0,1]) : \| f \|_\infty \leq n \} \).

Recall that a subset of a topological space \( X \) is of first category if it can be expressed as the union of countably many nowhere dense subsets of \( X \). A set \( A \subset X \) is nowhere dense if for any open set \( V \) there exists an open set \( U \subset V \) such that \( U \cap A = \emptyset \).