Risk Theory Prelims for Actuarial Students
Wednesday, 21 August 2019
MONT 313, 9:00 am - 1:00 pm

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.

2. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.

3. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Consider two risk measures for a loss random variable X with $0 < q < 1$:

- $\text{VaR}_q(X)$ is the value-at-risk for a $100q\%$ level
- $\text{TVaR}_q(X)$ is the tail-value-at-risk at the $100q\%$ level

(a) Describe the four (4) properties of a coherent risk measure.

(b) Suppose X_1 and X_2 are independent and identically distributed random variables with

$$\Pr(X_i = 0) = 0.10, \qquad \Pr(X_i = 1) = 0.85, \qquad \Pr(X_i = 2) = 0.05,$$

for $i = 1, 2$.

(i) Find the cumulative distribution function (cdf) of X_i, for $i = 1, 2$.
(ii) Find the cumulative distribution function (cdf) of $X_1 + X_2$.
(iii) Calculate $\text{VaR}_{0.925}(X_i)$ for $i = 1, 2$ and $\text{VaR}_{0.925}(X_1 + X_2)$.
(iv) Calculate $\text{TVaR}_{0.925}(X_i)$ for $i = 1, 2$ and $\text{TVaR}_{0.925}(X_1 + X_2)$.

(c) Comment on the subadditivity property of value-at-risk and tail-value-at-risk based on the results in (b).

Question No. 2:

In a collective risk model where the aggregate claim is defined by $S = X_1 + X_2 + \cdots + X_N$, you are given:

(i) Claim frequency N has a Poisson distribution with mean 5.

(ii) Claim amount X has the distribution $p(1) = 0.4$, $p(2) = 0.5$, and $p(3) = 0.1$.

(a) Use the Panjer's recursion formula to show that

$$\Pr(S = n) = \frac{1}{n} \times \left[k_1 \Pr(S = n-1) + k_2 \Pr(S = n-2) + k_3 \Pr(S = n-3)\right],$$

for constants k_1, k_2, and k_3. Determine the values of these constants.

(b) Calculate $\mathrm{E}\left[(S-2)_+\right]$.

Question No. 3:

Suppose the aggregate loss S for a pool of insurance contracts is given by

$$S = \sum_{i=1}^{N} X_i$$

where N is the claim frequency and X_i are the claim severities. Assume that:

- N has a Poisson distribution with mean $\lambda = 100$ and
- X_i has an exponential distribution with mean $\alpha = 10$ for $i = 1, 2, \ldots$.

Now all the policies in the pool are modified with a deductible of $d = 1$ and a maximum covered loss of $u = 100$. Denote the aggregate loss after modificiations by

$$\tilde{S} = \sum_{i=1}^{\tilde{N}} \tilde{X}_i$$

where $\tilde{N}$ is the claim frequency after modifications and $\tilde{X}_i$ are the claim severities after modifications.

(a) Show that $\tilde{N}$ has a Poisson distribution and calculate its mean $\tilde{\lambda} = \mathrm{E}[\tilde{N}]$.

(b) Calculate the mean and second moment of $\tilde{X}$: $\mathrm{E}[\tilde{X}]$ and $\mathrm{E}[\tilde{X}^2]$

(c) Calculate the mean and variance of $\tilde{S}$: $\mathrm{E}[\tilde{S}]$ and $\mathrm{Var}[\tilde{S}]$

Question No. 4:

For any distribution of X with a non-zero mean μ and a standard deviation σ, the ratio σ/μ is called the coefficient of variation of X. Denote this by $\mathrm{CV}(X)$.

(a) Prove or disprove: $\mathrm{CV}(X) = \mathrm{CV}(aX)$ for any non-zero real number a.

(b) For a sequence of random variables X_i, for $i = 1, 2, \ldots, n$, with aggregate sum $S = X_1 + \cdots + X_n$:

   (i) Can you find a sufficient condition for $\mathrm{CV}(S) = \sum_{i=1}^{n} \mathrm{CV}(X_i)$?

(ii) In the case where X_i's are i.i.d., explain why the coefficient of variation of S diminishes to zero as $n \to \infty$.

(c) For a Gamma distribution with scale parameter a:

(i) Find expression for the coefficient of variation and show that it does not depend on a.

(ii) What can you say about the Gamma distribution when its coefficient of variation is equal to 1?

(d) A random variable X is said to be log-normal if $\log(X)$ has a normal distribution with mean μ and variance σ^2. Show that if k is the coefficient of variation of X, then

$$\mathrm{E}[X] = e^{\mu}\sqrt{1+k^2} \quad \text{and} \quad \mathrm{Var}[X] = e^{2\mu}(1+k^2)k^2.$$

Question No. 5:

Losses X follow a Pareto(α, θ) distribution. An insurance company offers two types of policies to cover losses X: Type Q and Type R.

(i) Type Q has an ordinary deductible of d with no policy limit.

(ii) Type R has a policy limit of u but with no deductible.

(a) Find an expression of deductible d in terms of u such that both Type Q and Type R have the same expected cost per loss.

(b) Suppose $\alpha = 3$ and $\theta = 2000$. An insured chooses another type of policy that has both a deductible of 500 and a policy limit of 5000. Calculate the expected cost per loss for this coverage.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with scale parameter $a > 0$ if its density has the form

$$f(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad \text{for } x > 0.$$

A random variable X is said to be Pareto(α, θ) if its cumulative distribution function is expressed as

$$F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}.$$

A discrete random variable N is said to belong to the $(a, b, 0)$ class of distributions if it satisfies the relation

$$\Pr(N = k) = p_k = \left(a + \frac{b}{k}\right) \cdot p_{k-1}, \quad \text{for } k = 1, 2, \ldots,$$

for some constants a and b. The initial value p_0 is determined so that $\sum_{k=0}^{\infty} p_k = 1$.