COMPLEX ANALYSIS PRELIM

JANUARY, 2020

Notation and conventions:
- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

Problem 1. Let f be an entire function. Assume

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \le r^{20} \quad \text{for all } r \ge 100.$$

Show that f is a polynomial with degree ≤ 20.

Problem 2. How many zeros counting multiplicities does the polynomial

$$p(z) = 2z^5 + z^4 + 7z^2 + 2$$

have in the region $\{z \in \mathbb{C}; 1 < |z| < 2\}$? Prove your assertion.

Problem 3. Let $G = \{re^{i\theta} \mid 0 < r < 1, -\pi/2 < \theta < \pi\}$. Explicitly describe a one-to-one conformal map of G onto the unit disk $\mathbb{D}$, by using an explicit formula.

Problem 4. Let $\mathbb{D}^* = \mathbb{D} \setminus \{0\} = \{0 < |z| < 1\}$. Find the holomorphic automorphism group $\mathrm{Aut}(\mathbb{D}^*)$ (i.e., find all biholomorphic maps from $\mathbb{D}^*$ onto itself). Prove your assertion.

Problem 5. Let C_R be the *lower* semi-circle of radius $R > 0$, i.e., $C_R = \{Re^{i\theta}; \pi \le \theta \le 2\pi\}$ with positive direction. Compute the limit

$$\lim_{R \to +\infty} \int_{C_R} \frac{e^{iz}}{z} dz.$$

Problem 6. Let f be a holomorphic function on $\mathbb{D}$. Assume $f(0) = 0$ and $\mathrm{Re} f \le A$ on $\mathbb{D}$ for some constant $A > 0$. Show that

$$|f(z)| \le \frac{2A|z|}{1 - |z|} \quad \text{for all } z \in \mathbb{D}.$$

Problem 7. Is there a holomorphic function g defined on $\Omega = \{z \in \mathbb{C}; |z| > 200\}$ such that

$$g'(z) = \frac{z^{51}}{\prod_{m=1}^{100}(z - m)} \quad \text{for all } z \in \Omega?$$

Prove your assertion.