An introduction to ordinal numbers:
an excerpt from my DRP with Noah Hughes

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A pack of wolves, a bunch of grapes or a flock of pigeons are all examples of sets of things.
Definition: A relation on a set $A$ is a set of ordered pairs of elements of $A$. Alternatively, you can think of it as a subset of the Cartesian product $A \times A$. 
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If an ordered pair $(a, b)$ is an element of a relation $R$, we write $aRb$. Relations are almost always denoted by symbols; one prominent example being the equivalence relation $\equiv$. 
Ordering

**Definition:** An ordering is a relation that shows a hierarchy between elements of a set.

**Examples:** \((\mathbb{N}, \leq), (\mathbb{Q}, >), (\mathcal{P}(A), \subseteq)\).
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Definition: We call a set $A$ with an order $\leq$ partially ordered if for every $x, y,$ and $z$ in $A$, we have

$x \leq x;$

$x \leq y$ and $y \leq z$ implies $x \leq z;$

$x \leq y$ and $y \leq x$ implies $x = y;$

and totally ordered if in addition for every $x, y \in A$ either

$x \leq y$ or $y \leq x.$
Well Ordering

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$(\varnothing(A), \subseteq)$ is not a well order.
**Well Ordering**

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$(\wp(A), \subseteq)$ is not a well order.

**Example:** The set $\mathbb{Z}$, with the ordering $\leq$ (here in the usual sense), is totally ordered, but **not** well-ordered.  
Why?
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(\( \mathcal{P}(A), \subseteq \)) is not a well order.

Example: The set \( \mathbb{Z} \), with the ordering \( \leq \) (here in the usual sense), is totally ordered, but not well-ordered.

Why? Consider the set of negative integers.
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**Example:** The set $\mathbb{Z}$, with the ordering $\leq$ (here in the usual sense), is totally ordered, but **not** well-ordered. **Why?** Consider the set of negative integers.

We can, however, define a new order, $\leq_w$, that is a well ordering of the integers; we proceed to do so as an exercise in well ordering.
Example: Well Ordering of the Integers

Define $\leq_w$ such that, for all integers $x$, $y$, and $z$,

- if $|x| < |y|$, then $x \leq_w y$ (and vice-versa);
- if $|x| = |y|$, then
  - if $x < y$, then $x <_w y$ (and vice-versa);
  - if $x = y$, then $x =_w y$.

Our new $\leq_w$ is a well ordering of $\mathbb{Z}$. Why?
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More Well Orderings of $\mathbb{Z}$

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(\mathbb{Z}, \leq_{w+3}) \cong 2, -2, 3, -3, ..., 0, -1, 1
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Proof. Too long and technical for the scope of this presentation. We will see its impact soon, but first we turn to constructing ordinals.
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We denote this set as $\omega$, though you may know it as $\mathbb{N}$. 

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**Ordinals**
Constructing Finite Ordinals

$0 = \emptyset$

Note: For finite ordinals, $x < y$ implies $x \in y$ and $x \leq y$ implies $x \subseteq y$. 
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\[ 1 = \{0\} = \{\emptyset\} \]
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\begin{align*}
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\[ \vdots \]
\[ n = \{0, 1, 2, \ldots, n - 1\} \]
\[ n + 1 = \{0, 1, 2, \ldots, n - 1, n\} \]
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**Note:** For finite ordinals, \( x < y \) implies \( x \in y \) and \( x \leq y \) implies \( x \subseteq y \).
**Definition:** \( \omega \) is the set of all finite ordinals. In other words,

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$\omega$ is the first transfinite ordinal as well as the first limit ordinal.

Definition: A limit ordinal is any ordinal that has no immediate predecessor.
Transfinite Ordinals, continued

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Comparing to our well orderings of $\mathbb{Z}$

Recall:

$$ (\mathbb{Z}, \leq_w) \cong 0, -1, 1, -2, 2, ... $$

$$ (\mathbb{Z}, \leq_{w+1}) \cong -1, 1, -2, 2, ..., 0 $$

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Closing Thoughts

- Ordinal arithmetic
- Cardinal numbers and arithmetic
- Continuum hypothesis
Thank you!