Abstract Algebra Prelim Aug. 2020

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) The commutator subgroup of a group G, denoted by G', is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in G$.

   For an integer $n \geq 2$, define

   $$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbf{Z}/n\mathbf{Z})^\times,\ b \in \mathbf{Z}/n\mathbf{Z} \right\} \subset \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z}).$$

   (a) (**4 pts**) Show $N := \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbf{Z}/n\mathbf{Z}\}$ is a normal subgroup of G and $G/N \cong (\mathbf{Z}/n\mathbf{Z})^\times$.

   (b) (**2 pts**) Show N is cyclic with generator $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

   (c) (**4 pts**) When $n \geq 3$ is *odd*, use (a) and (b) to show that $G' = N$.

2. (**10 pts**) Let G be a group. Recall that an *automorphism* of G is an isomorphism $G \to G$. The two parts below can be done independently.

   (a) (**5 pts**) For a prime p, let G be a cyclic group of order p written multiplicatively. Prove that the automorphisms of G are precisely the functions $f\colon G \to G$ where $f(g) = g^k$ for all $g \in G$ and some exponent $k \not\equiv 0 \bmod p$.

   (b) (**5 pts**) For a prime p, let $G = (\mathbf{Z}/p\mathbf{Z})^2$. Show that the automorphisms of G are precisely the functions $f\colon G \to G$ where $f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+by \\ cx+dy \end{smallmatrix}\right)$ for some $a, b, c, d \in \mathbf{Z}/p\mathbf{Z}$ such that $ad - bc \not\equiv 0 \bmod p$.

3. (**10 pts**) Fix a group H and a group automorphism $f \in \mathrm{Aut}(H)$. Let $\varphi\colon \mathbf{Z} \to H$ by $\varphi(n) = f^n$. The two parts below about $H \rtimes_\varphi \mathbf{Z}$ can be done independently.

   (a) (**5 pts**) In $H \rtimes_\varphi \mathbf{Z}$, prove (h, n) has finite order if and only if h has finite order and $n = 0$.

   (b) (**5 pts**) Let $\psi\colon \mathbf{Z} \to H$ by $\psi(n) = f^{-n}$. Prove that $H \rtimes_\varphi \mathbf{Z} \cong H \rtimes_\psi \mathbf{Z}$ as groups.

4. (**10 pts**) For a finite subset $\{r_1, \ldots, r_k\}$ of $\mathbf{Q}$, the ring $\mathbf{Z}[r_1, \ldots, r_k]$ is the set of all sums of products of nonnegative powers of $r_1, \ldots, r_k$ with coefficients in $\mathbf{Z}$.

   (a) (**3 pts**) For relatively prime integers a and b, with $b \neq 0$, prove $\mathbf{Z}[a/b] = \mathbf{Z}[1/b]$.

   (b) (**3 pts**) Let $N > 1$ in $\mathbf{Z}$ have prime factorization $p_1^{e_1} \cdots p_k^{e_k}$ for distinct primes $p_1, \ldots, p_k$ and $e_j > 0$. Prove that $\mathbf{Z}[1/N] = \{a/N^e : a \in \mathbf{Z}, e \geq 0\}$ and $\mathbf{Z}[1/N] = \mathbf{Z}[1/p_1, \ldots, 1/p_k]$.

   (c) (**4 pts**) Prove that $\mathbf{Z}[1/N]$ has unit group $\langle -1, p_1, \ldots, p_k \rangle = \{\pm p_1^{a_1} \cdots p_k^{a_k} : a_j \in \mathbf{Z}\}$.

5. (**10 pts**) Let A be an $n \times n$ matrix with complex entries.

   (a) (**4 pts**) Show that A and its transpose $A^\top$ have the same eigenvalues in $\mathbf{C}$.

   (b) (**6 pts**) If λ is an eigenvalue of A, with $A\mathbf{v} = \lambda\mathbf{v}$ and $A^\top\mathbf{w} = \lambda\mathbf{w}$ in $\mathbf{C}^n$, then prove that the matrix $B = \mathbf{v}\mathbf{w}^\top$ commutes with A. (Consider $\mathbf{v}$ and $\mathbf{w}$ as column vectors, so $\mathbf{v}\mathbf{w}^\top$ is an $n \times n$ matrix. You may assume $\mathbf{v}$ and $\mathbf{w}$ are not $\mathbf{0}$, but it doesn't matter.)

6. (**10 pts**) Give examples as requested, with justification.

   (a) (**2.5 pts**) A prime factorization of $3 + i$ in $\mathbf{Z}[i]$.

   (b) (**2.5 pts**) A unit in $\mathbf{Z}[x]/(x^2)$ other than ± 1.

   (c) (**2.5 pts**) A maximal ideal $\mathfrak{m}$ in $\mathbf{Z}[x]$ such that $\mathbf{Z}[x]/\mathfrak{m}$ has order 27.

   (d) (**2.5 pts**) A ring-theoretic property that $\mathbf{Q}[x]/(x^2 - 1)$ and $\mathbf{Q}[x]/(x^2)$ do *not* share (so the rings are not isomorphic).