Justify all your steps. You may use any results that you know unless the question says otherwise, but don’t invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (10 pts) The commutator subgroup of a group $G$, denoted by $G'$, is the subgroup generated by all commutators $[x,y] = xyx^{-1}y^{-1}$ for $x, y \in G$.

For an integer $n \geq 2$, define

$$G = \left\{ \left( \begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) : a \in (\mathbb{Z}/n\mathbb{Z})^\times, \ b \in \mathbb{Z}/n\mathbb{Z} \right\} \subset \text{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

(a) (4 pts) Show $N := \left\{ \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) : b \in \mathbb{Z}/n\mathbb{Z} \right\}$ is a normal subgroup of $G$ and $G/N \cong (\mathbb{Z}/n\mathbb{Z})^\times$.

(b) (2 pts) Show $N$ is cyclic with generator $\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$.

(c) (4 pts) When $n \geq 3$ is odd, use (a) and (b) to show that $G' = N$.

2. (10 pts) Let $G$ be a group. Recall that an automorphism of $G$ is an isomorphism $G \to G$.

The two parts below can be done independently.

(a) (5 pts) For a prime $p$, let $G$ be a cyclic group of order $p$ written multiplicatively. Prove that the automorphisms of $G$ are precisely the functions $f : G \to G$ where $f(g) = g^k$ for all $g \in G$ and some exponent $k \not\equiv 0 \mod p$.

(b) (5 pts) For a prime $p$, let $G = (\mathbb{Z}/p\mathbb{Z})^2$. Show that the automorphisms of $G$ are precisely the functions $f : G \to G$ where $f(a,b) = (a^e, b^f)$ for some $a, b \in \mathbb{Z}/p\mathbb{Z}$ such that $ad - bc \not\equiv 0 \mod p$.

3. (10 pts) Fix a group $H$ and a group automorphism $f \in \text{Aut}(H)$. Let $\varphi : \mathbb{Z} \to H$ by $\varphi(n) = f^n$.

The two parts below about $H \rtimes_{\varphi} \mathbb{Z}$ can be done independently.

(a) (5 pts) In $H \rtimes_{\varphi} \mathbb{Z}$, prove $(h,n)$ has finite order if and only if $h$ has finite order and $n = 0$.

(b) (5 pts) Let $\psi : \mathbb{Z} \to H$ by $\psi(n) = f^{-n}$. Prove that $H \rtimes_{\psi} \mathbb{Z} \cong H \rtimes_{\varphi} \mathbb{Z}$ as groups.

4. (10 pts) For a finite subset $\{r_1, \ldots, r_k\}$ of $\mathbb{Q}$, the ring $\mathbb{Z}[r_1, \ldots, r_k]$ is the set of all sums of products of nonnegative powers of $r_1, \ldots, r_k$ with coefficients in $\mathbb{Z}$.

(a) (3 pts) For relatively prime integers $a$ and $b$, with $b \neq 0$, prove $\mathbb{Z}[a/b] = \mathbb{Z}[1/b]$.

(b) (3 pts) Let $N > 1$ in $\mathbb{Z}$ have prime factorization $p_1^{e_1} \cdots p_k^{e_k}$ for prime primes $p_1, \ldots, p_k$ and $e_j > 0$. Prove that $\mathbb{Z}[1/N] = \{a/N^e : a \in \mathbb{Z}, e \geq 0\}$ and $\mathbb{Z}[1/N] = \mathbb{Z}[1/p_1, \ldots, 1/p_k]$.

(c) (4 pts) Prove that $\mathbb{Z}[1/N]$ has unit group $\langle -1, p_1, \ldots, p_k \rangle = \{\pm p_1^{a_1} \cdots p_k^{a_k} : a_j \in \mathbb{Z}\}$.

5. (10 pts) Let $A$ be an $n \times n$ matrix with complex entries.

(a) (4 pts) Show that $A$ and its transpose $A^\top$ have the same eigenvalues in $\mathbb{C}$.

(b) (6 pts) If $\lambda$ is an eigenvalue of $A$, with $Av = \lambda v$ and $A^\top w = \lambda w$ in $\mathbb{C}^n$, then prove that the matrix $B = \mathbf{v} \mathbf{w}^\top$ commutes with $A$. (Consider $\mathbf{v}$ and $\mathbf{w}$ as column vectors, so $\mathbf{v} \mathbf{w}^\top$ is an $n \times n$ matrix. You may assume $\mathbf{v}$ and $\mathbf{w}$ are not $\mathbf{0}$, but it doesn’t matter.)

6. (10 pts) Give examples as requested, with justification.

(a) (2.5 pts) A prime factorization of $3 + i$ in $\mathbb{Z}[i]$.

(b) (2.5 pts) A unit in $\mathbb{Z}[x]/(x^2)$ other than $\pm 1$.

(c) (2.5 pts) A maximal ideal $\mathfrak{m}$ in $\mathbb{Z}[x]$ such that $\mathbb{Z}[x]/\mathfrak{m}$ has order 27.

(d) (2.5 pts) A ring-theoretic property that $\mathbb{Q}[x]/(x^2 - 1)$ and $\mathbb{Q}[x]/(x^2)$ do not share (so the rings are not isomorphic).