Applied Math Prelim August 2020

1. Let H be a Hilbert space, $T : H \to H$ be a linear bounded operator and $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges weakly in H, i.e. $x_n \rightharpoonup x_0$.
   (a) Show that $Tx_n \rightharpoonup Tx_0$ weakly in H.
   (b) Show that if T is compact, then $Tx_n \to Tx_0$.
   (c) Suppose for every weakly convergence sequence $x_n \rightharpoonup x_0$, we always have $Tx_n \to Tx_0$. Show that T is compact. (Hint: every bounded sequence in H has a weakly convergence subsequence.)
   (d) Suppose $T_n : H \to H$ is linear bounded compactor operator for each $n = 1, 2, 3, \ldots$. Suppose $T_n \to T_0$ in operator norm. Show that T_0 is compact.

2. For any $f \in L^1_{loc}(\mathbb{R}^n)$, we let $\tilde{f} \in \mathcal{D}'(\mathbb{R}^n)$ be the distribution such that $\tilde{f}(\phi) = \int_{\mathbb{R}^n} f(z)\varphi(z)\, dz$ for any test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
   (a) Let $f \in L^1(\mathbb{R}^n)$ be non-negative with $\int_{\mathbb{R}^n} f(z)\, dz = 1$. Define $f_j(z) = j^n f(jz)$. Show that $\tilde{f}_j \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$, where δ is the Dirac distribution (delta function). Does your proof work if f can change sign?
   (b) Let $n = 2$ and $z = (x, y) \in \mathbb{R}^2$. Take

   $$f(x,y) = \begin{cases} x^2 + y^2, & \text{if } x^2 + y^2 \le 1, \\ 0, & \text{if otherwise.} \end{cases}$$

   Compute the distribution derivative $\partial_1 \tilde{f} = \frac{\partial}{\partial x}\tilde{f}$. Simplify as much as possible.
   (c) If $\varphi(x, y) = x$ when $x^2 + y^2 \le 2$ and has compact support in $\mathbb{R}^2$. Using (b) or otherwise, evaluate $(\partial_1 \tilde{f})(\varphi)$.

3. (a) Find the Green's function $G(x, y)$ for the operator A where

   $$Au \equiv u'' - u$$

   with $u(0) = u(\pi) = 0$.
   (b) Define $T : L^2(0, \pi) \to L^2(0, \pi)$ such that for any $f \in L^2(0, \pi)$,

   $$(Tf)(x) = \int_0^{\pi} G(x, y) f(y)\, dy \, .$$

   Explain what spectral theorem is and why it is applicable to T.

(c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.

(d) Find an orthonormal basis of $L^2(0, \pi)$ via operator A.

4. Let $f : D \to Y$ be a mapping from an open set D in a normed linear space X to another normed linear space Y.

   (a) State the definition of Fréchet derivative of f.

   (b) Let $D = X = Y = C[0, 1]$, the set of continuous functions equipped with the uniform norm. For any $u \in C[0, 1]$, define

   $$f(u)\,(t) = \sin(u(t)) \quad \text{for all } t \in [0, 1]\ .$$

   Show that $f : C[0, 1] \to C[0, 1]$ is Frechet differentiable at any $u \in C[0, 1]$ and its derivative is given by

   $$f'(u)h\ (t) = (\cos u(t))\ h(t)$$

   for all $h \in C[0, 1]$ and $t \in [0, 1]$.
   (Depending on how you prove this. You may or may not need $\sin(A + B) = \sin A \cos B + \cos A \sin B$.)

5. Let H be a Hilbert space and $A : H \to H$ be a linear bounded compact operator. Show that the range of $I + A$ is closed.