1. Let $H$ be a Hilbert space, $T : H \to H$ be a linear bounded operator and $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges weakly in $H$, i.e. $x_n \rightharpoonup x_0$.

(a) Show that $Tx_n \rightharpoonup Tx_0$ weakly in $H$.

(b) Show that if $T$ is compact, then $Tx_n \to Tx_0$.

(c) Suppose for every weakly convergence sequence $x_n \rightharpoonup x_0$, we always have $Tx_n \to Tx_0$. Show that $T$ is compact. (Hint: every bounded sequence in $H$ has a weakly convergence subsequence.)

(d) Suppose $T_n : H \to H$ is linear bounded compactor operator for each $n = 1, 2, 3, \ldots$. Suppose $T_n \to T_0$ in operator norm. Show that $T_0$ is compact.

2. For any $f \in L^1_{loc}(\mathbb{R}^n)$, we let $\tilde{f} \in D'(\mathbb{R}^n)$ be the distribution such that $\tilde{f}(\phi) = \int_{\mathbb{R}^n} f(z) \phi(z) \, dz$ for any test function $\phi \in D(\mathbb{R}^n)$.

(a) Let $f \in L^1(\mathbb{R}^n)$ be non-negative with $\int_{\mathbb{R}^n} f(z) \, dz = 1$. Define $f_j(z) = \frac{1}{j^n} f(jz)$. Show that $f_j \to \delta$ in $D'(\mathbb{R}^n)$, where $\delta$ is the Dirac distribution (delta function). Does your proof work if $f$ can change sign?

(b) Let $n = 2$ and $z = (x, y) \in \mathbb{R}^2$. Take $f(x, y) = \begin{cases} x^2 + y^2, & \text{if } x^2 + y^2 \leq 1, \\ 0, & \text{if otherwise.} \end{cases}$

Compute the distribution derivative $\partial_1 \tilde{f} = \frac{\partial}{\partial x} \tilde{f}$. Simplify as much as possible.

(c) If $\phi(x, y) = x$ when $x^2 + y^2 \leq 2$ and has compact support in $\mathbb{R}^2$. Using (b) or otherwise, evaluate $(\partial_1 \tilde{f})(\phi)$.

3. (a) Find the Green’s function $G(x, y)$ for the operator $A$ where $Au \equiv u'' - u$

with $u(0) = u(\pi) = 0$.

(b) Define $T : L^2(0, \pi) \to L^2(0, \pi)$ such that for any $f \in L^2(0, \pi)$,

$$(Tf)(x) = \int_0^\pi G(x, y) f(y) \, dy .$$

Explain what spectral theorem is and why it is applicable to $T$. 

(c) Show that \( \|T\| = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } T \} \).

(d) Find an orthonormal basis of \( L^2(0, \pi) \) via operator \( A \).

4. Let \( f : D \to Y \) be a mapping from an open set \( D \) in a normed linear space \( X \) to another normed linear space \( Y \).

(a) State the definition of Fréchet derivative of \( f \).

(b) Let \( D = X = Y = C[0, 1] \), the set of continuous functions equipped with the uniform norm. For any \( u \in C[0, 1] \), define

\[
f(u)(t) = \sin(u(t)) \quad \text{for all } t \in [0, 1].
\]

Show that \( f : C[0, 1] \to C[0, 1] \) is Frechet differentiable at any \( u \in C[0, 1] \) and its derivative is given by

\[
f'(u)h(t) = (\cos u(t)) \, h(t)
\]

for all \( h \in C[0, 1] \) and \( t \in [0, 1] \).

(Depending on how you prove this. You may or may not need
\[
\sin(A + B) = \sin A \cos B + \cos A \sin B.
\]

5. Let \( H \) be a Hilbert space and \( A : H \to H \) be a linear bounded compact operator. Show that the range of \( I + A \) is closed.