Complex Analysis Prelim

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Instructions: Do as many of the following problems as you can. Four completely correct answers will guarantee a Ph.D. pass. Incomplete solutions or solutions with a substantial error may be assigned no credit. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill in the gap. You may use any standard theorem from the complex analysis course, identifying it either by name or stating it in full.

Notation and Conventions:

- $\mathbb{C}$ denotes the complex numbers, $\mathbb{D}$ denotes the open unit disk, and $\mathbb{Z}$ denotes the integers.
- $D(z,r) = \{w \in \mathbb{C} : |w-z| < r\}$ denotes the open disk with center $z \in \mathbb{C}$ and radius $r > 0$.
- The terminology *analytic function* and *holomorphic function* may be used interchangeably.

Problems:

1. Let $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ be a formal power series with complex coefficients. Prove that if the series converges for every $z \in \mathbb{D}$, then f is analytic in $\mathbb{D}$.

2. Describe all analytic functions $f : D(4,3) \to \mathbb{C}$ such that $e^{f(z)} = z$ for all $z \in D(4,3)$. For each such function, find $\inf_{z \in D(4,3)} |f(z)|$. Prove that your answer is correct.

3. Let γ be a closed C^1 curve in $\mathbb{C} \backslash \mathbb{D}$ that winds around the origin twice in the counterclockwise direction. Compute $\displaystyle\int_\gamma \frac{8z^2 - 6z + 1}{6z^2 - 5z + 1}\, dz$. As always, justify your computation.

4. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \setminus \{x + 0i : x \in (0,1]\}$. Find a one-to-one analytic function $f : \mathbb{D} \to \Omega$ such that $f(\mathbb{D}) = \Omega$, $f(0) = 2$, and $f'(0) > 0$. You may describe f using a composition of maps.

5. Let $\Omega \subseteq \mathbb{C}$ be a connected, open set. Suppose that $f, f_1, f_2, \cdots : \Omega \to \mathbb{C}$ are analytic functions and $f_n \to f$ converges uniformly on compact sets. Prove that $f_n' \to f'$ uniformly on compact sets.

6. Let $\Omega \subseteq \mathbb{C}$ be a connected, open set and let $f : \Omega \to \mathbb{C}$ be an analytic function. Suppose that the closed disk $\overline{D(z,r)} \subseteq \Omega$ for some $z \in \Omega$ and $r > 0$ and

$$\delta = \inf_{\zeta \in \partial D(z,r)} |f(\zeta) - f(z)| > 0.$$

   Prove that $D(f(z), \delta/2) \subseteq f(D(z,r))$.

7. Prove that there exists a sequence $p_n : \mathbb{C} \to \mathbb{C}$ of polynomials such that

$$q(z) = \lim_{n\to\infty} p_n(z)$$

   exists for all $z \in \mathbb{C}$ and $q(\mathbb{C}) = \mathbb{Z}$.