Probability Prelim Exam for Actuarial Students
August 2020

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Let X be a random variable defined on a probability space $(\Omega, \mathscr{F}, P)$. Suppose that $P(X > 0) > 0$. Show that there exits a $\delta > 0$ such that $P(X \geq \delta) > 0$.

2. (10 points) Let $(\Omega, \mathcal{F})$ be a measurable space and let $X_1, X_2, \ldots$ be real-valued random variables. Show that the set $\{\omega \in \Omega : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n(\omega) > 1\}$ is an event.

3. (10 points) State and prove the (first) Borel-Cantelli lemma.

4. (20 points) Prove or disprove the following

   (a). If $X_1, X_2, \ldots$ in $L^1(P)$, and $M_n = \max\{X_1, \ldots, X_n\}$, then $M_n \in L^1(P)$.

   (b). Suppose (X_n) are random variables satisfying

   $$E[X_n] = 2 \text{ and } E[X_n^2] = 4 + \frac{1}{n},$$

   then $X_n \to 2$ in probability.

   (c). If X is a random variable in $L^1(P)$, then $\lim_{x\to\infty} xP(|X| > x) = 0$.

   (d). If X is a random variable satisfying $\lim_{x\to\infty} xP(|X| > x) = 0$, then X is in $L^1(P)$.

5. (10 points) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion and let

   $$Z_n = \frac{1}{n} \sum_{i=1}^{n} B_i,$$

   where n is a positive integer. Compute the distribution of Z_n.

6. (10 points) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion and let

   $$\tau = \inf\{t \geq 0 : B_t \leq -1 \text{ or } B_t \geq 2\}.$$

   (a). Compute the distribution of B_τ.

   (b). Calculate $E[\tau]$.