Real Analysis Preliminary Exam, August 2020

Instructions and notation:

(i) All problems are worth 10 points. Give full justifications for all answers in the exam booklet.

(ii) Lebesgue measure on $\mathbb{R}^n$ is denoted by $|\cdot|$.

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1. Assume that f is a Lebesgue integrable function in $\mathbb{R}^n$. If $\int_E f = 0$ for every Lebesgue measurable set E in $\mathbb{R}^n$, then $f = 0$ a.e in $\mathbb{R}^n$

2. Let (X, Σ, μ) be a measure space. Let $\{f_k\}_{k\in\mathbb{N}}$ and $\{\phi_k\}_{k\in\mathbb{N}}$ be sequences of measurable functions. Assume that $f_k \to f$ pointwise, $\phi_k \to \phi$ μ-a.e. and $|f_k| \leq \phi_k$ μ-a.e. If $\phi \in L^1(X, d\mu)$ and $\int \phi_k \to \int \phi$, then $\int |f_k - f| \to 0$.

3. Let (X, Σ, μ) be a measure space and suppose that $f, \{f_k\}_{k\in\mathbb{N}} \in L^p(X, d\mu)$ for $0 < p \leq \infty$.
   (a) (4 points) Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$.
   (b) (4 points) Conversely, if $f_k \to f$ pointwise and $\|f_k\|_p \to \|f\|_p$ for $0 < p < \infty$, show that $\|f - f_k\|_p \to 0$.
   (c) (2 points) Show that (b) may fail if $p = \infty$.

4. Suppose that W is a Lebesgue nonmeasurable set in $[0, 1]$. Prove that there exists some $0 < \epsilon < 1$ such that for any Lebesgue measurable subset $E \subset [0, 1]$ with $|E| \geq \epsilon$, the set $W \cap E$ must be Lebesgue nonmeasurable.

5. (a) (5 points) Let $(X, \Sigma, \mu), (X, \Sigma, \nu)$ be two measure spaces such that $\nu(X) < \infty$. Prove that ν is absolutely continuous with respect to μ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$ then $\nu(A) < \epsilon$.
   (b) (5 points) Give an example of a pair of measure spaces $(X, \Sigma, \mu), (X, \Sigma, \nu)$ such that ν is absolutely continuous with respect to μ, but given $\epsilon > 0$ there is no $\delta > 0$ such that $\nu(A) < \epsilon$ for every $A \in \Sigma$ with $\mu(A) < \delta$.

6. Assume that $\{E_1, \ldots, E_m\}$ is a family of Lebesgue measurable subsets of $\mathbb{R}^n$ and $k > 0$ is a positive integer. Let $E \subset \mathbb{R}^n$ be a measurable subset with $|E| > 0$. Suppose that almost every $x \in E$ belongs to at least k of the E_j. Prove that there is at least one E_l such that $|E_l| \geq \frac{k}{m}|E|$.

7. (a) (3 points) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with $|E| > 0$. Use the Lebesgue Differentiation Theorem to prove that
   $$\lim_{r\to 0} \frac{|Q(x, r) \cap E|}{|Q(x, r)|} = 1 \text{ for a.e. } x \in E,$$
   where $Q(x, r)$ denotes a cube with sides parallel to the axes, centered at x with sidelength $2r$.
   (b) (7 points) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with $|E| > 0$ and let D be a dense countable subset of $\mathbb{R}^n$. Prove that $|\mathbb{R}^n \setminus \cup_{x\in D} (x + E)| = 0$, where $x + E := \{x + e : e \in E\}$. *Hint: Use part (a).*