Fuzzy Logic

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In Mathematics, the **set** is a rudimentary concept. From sets, or **collections of objects**, we can make statements about the objects within them, connections between sets, and even create mappings from one set to another.
Objects are related to classical sets by a clear, crisp sense of membership. For some object $x$, there are only two possibilities:

\[ x \in A \]

or

\[ x \notin A. \]

When dealing with the abstract, classical sets serve their purpose. But in the real-world (as we all know too well) there is uncertainty and ambiguity.
Fuzzy sets are actually generalizations of classical sets, which are defined by the characteristic function $\chi_A(x)$

$$\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}$$
A fuzzy set is a class (collection of sets or objects) with a continuum of membership grades.

**Definition**

A fuzzy set $A$ (subset of $X$) is defined as a mapping

$$A : X \rightarrow [0, 1]$$

where $A(x)$ is the membership degree of $x$ to the fuzzy set $A$. We denote by $\mathcal{F}$ the collection of all fuzzy subsets of $X$.
Fuzzy sets allow for a robust definition of sets or categories that can handle the obscurity that comes with language and physical systems.

Ex: To what extent are certain car speeds ”slow”, ”average”, ”fast”, etc.?
"very slow" (black): $A(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 10 \\
\frac{20-x}{10} & \text{if } 10 \leq x < 20 \\
0 & \text{if } 20 \leq x \leq 60 
\end{cases}$
Connectives

Definition (Intersection)
Let $A, B \in \mathcal{F}(X)$. The **intersection** of $A$ and $B$ is the fuzzy set $C$ with

$$C(x) = \min \{A(x), B(x)\} = A(x) \land B(x), \forall x \in X.$$ 

We denote $C = A \land B$.

Definition (Union)
Let $A, B \in \mathcal{F}(X)$. The **union** of $A$ and $B$ is the fuzzy set $C$ with

$$C(x) = \max \{A(x), B(x)\} = A(x) \lor B(x), \forall x \in X.$$ 

We denote $C = A \lor B$.

Definition (Complementation)
Let $A \in \mathcal{F}(X)$ be a fuzzy set. The **complement** of $A$ is the fuzzy set where

$$B(x) = 1 - A(x), \forall x \in X.$$
Fuzzy Relations

Definition (Classical Relation)

A subset $R \subseteq X \times Y$ where $X$ and $Y$ are classical sets, is a classical relation.
Similar to classical sets, a classical relation can be characterized by a function $R : X \times Y \rightarrow \{0, 1\}$,

$$R(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

Definition (Fuzzy Relation)

Let $X, Y$ be two classical sets. A mapping $R : X \times Y \rightarrow [0, 1]$ is called a fuzzy relation. The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between $x$ and $y$. 
Fuzzy Relations

If $X$ and $Y$ are finite sets such that $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ a fuzzy relation between the two sets can be represented as the following matrix:

$$
R = \begin{pmatrix}
R(x_1, y_1) & R(x_1, y_2) & \cdots & R(x_1, y_n) \\
R(x_2, y_1) & R(x_2, y_2) & \cdots & R(x_2, y_n) \\
\vdots & \vdots & \ddots & \vdots \\
R(x_m, y_1) & R(x_m, y_2) & \cdots & R(x_m, y_n)
\end{pmatrix}
$$
Max-Min Compositions

Definition

Let $R \in \mathcal{F}(X \times Y)$ and $S \in \mathcal{F}(Y \times Z)$ be fuzzy relations. Then $R \circ S(x, z) \in \mathcal{F}(X \times Z)$, defined as

$$R \circ S(x, z) = \bigvee_{y \in Y} R(x, y) \land S(y, z),$$

is the max-min composition of the fuzzy relations $R$ and $S$. 

Max-Min Compositions for Finite Sets

Let \( X = \{x_1, ..., x_n\} \), \( Y = \{y_1, ..., y_m\} \), \( Z = \{z_1, ..., z_p\} \) be finite sets. If \( R = (r_{ij})_{i=1,...,n,j=1,...,m} \in \mathcal{F}(X \times Y) \) and \( S = (s_{jk})_{j=1,...,n,k=1,...,p} \in \mathcal{F}(X \times Y) \) are discrete fuzzy relations then the composition \( T = (t_{ik})_{j=1,...,n,k=1,...,p} = R \circ S \in \mathcal{F}(X \times Z) \) is given by

\[
t_{ik} = \bigvee_{j=1}^{m} r_{ij} \land s_{jk},
\]

\( i = 1, ..., n, k = 1, ..., p \)

Example: If \( R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix} \) and \( S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix} \) then

\[
R \circ S = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}
\]
Fuzzy logic systems have been used for modelling diagnosis in medicine.

The max-min composition allows for a mapping from a set of patients to a set of symptoms, then from symptoms to a set of diagnoses.[Samuel and Balamurugan, 2012]
A function $N : [0, 1] \rightarrow [0, 1]$ is called a negation if $N(0) = 1$, $N(1) = 0$ and $N$ is non-increasing ($x < y \implies N(x) > N(y)$). A negation is called a strict negation if it is strictly decreasing ($x < y \implies N(x) > N(y)$) and continuous. A strict negation is said to be a strong negation if $N(N(x)) = x$. 
Triangular norms and conorms are generalizations of the basic connectives of fuzzy sets.

**Definition**

Let $T, S : [0, 1]^2 \rightarrow [0, 1]$. Consider the following properties:

- **$T_1$**: $T(x, 1) = x$ (identity)
- **$S_1$**: $S(x, 0) = x$
- **$T_2$**: $T(x, y) = T(y, x)$ (commutativity)
- **$S_2$**: $S(x, y) = S(y, x)$
- **$T_3$**: $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)
- **$S_3$**: $S(x, S(y, z)) = S(S(x, y), z)$
- **$T_4$**: If $x \leq u$ and $y \leq v$ then $T(x, y) \leq T(u, v)$ (monotonicity)
- **$S_4$**: If $x \leq u$ and $y \leq v$ then $S(x, y) \leq S(u, v)$

A **triangular norm** (t-norm) is a function $T : [0, 1]^2 \rightarrow [0, 1]$ that satisfies $T_1 - T_4$

A **triangular conorm** (t-conorm) is a function $S : [0, 1]^2 \rightarrow [0, 1]$ that satisfies $S_1 - S_4$
Ex: Gödel, Gougen t-norm and t-conorm

Gödel t-norm, t-conorm, standard negation

\[ x \land y = \min \{x, y\} \]
\[ x \lor y = \max \{x, y\} \]
\[ N(x) = 1 - x \]

\[ T_1 : \min \{x, 1\} = x \]
\[ S_1 : \max \{x, 0\} = x \]
\[ T_2 : \min \{x, y\} = \min \{y, x\} \]
\[ S_2 : \max \{x, y\} = \max \{y, x\} \]
\[ T_3 : \min \{x, \min \{y, z\}\} = \min \{\min \{x, y\}, z\} \]
\[ S_3 : \max \{x, \max \{y, z\}\} = \max \{\max \{x, y\}, z\} \]
\[ T_4 : \text{If } x \leq u \text{ and } y \leq v \text{ then } \min \{x, y\} \leq \min \{u, v\} \]
\[ S_4 : \text{If } x \leq u \text{ and } y \leq v \text{ then } \max \{x, y\} \leq \max \{u, v\} \]
Gödel, Gougen t-norm and t-conorm

Gougen t-norm, t-conorm, standard negation

\[ x_{T_G} y = x \cdot y \]
\[ x_{S_G} y = x + y - xy \]
\[ N(x) = 1 - x \]

\[ T_1 : x \cdot 1 = x \]
\[ S_1 : x + 0 - 0 = x \]
\[ T_2 : x \cdot y = y \cdot x \]
\[ S_2 : x + y - xy = y + x - yx \]
\[ T_3 : x \cdot (yz) = (xy) \cdot z \]
\[ S_3 : x + (y + z - yz) - x(y + z - yz) = (x + y - xy) + z - z(x + y - xy) \]
\[ T_4 : \text{If } x \leq u \text{ and } y \leq v \text{ then } x \cdot y \leq u \cdot v \]
\[ S_4 : \text{If } x \leq u \text{ and } y \leq v \text{ then } x + y - xy \leq u + v - uv \]
DeMorgan Triplets

A triplet \((S, T, N)\) is called a **De Morgan triplet** if \(T\) is a t-norm, \(S\) is a t-conorm, \(N\) is a strong negation, and if they fulfill De Morgan’s law

\[
S(x, y) = N(T(N(x), N(y)))
\]

**Example 2.15.** The minimum, maximum, and standard negation

\[
x \land y = \min \{x, y\}
\]
\[
x \lor y = \max \{x, y\}
\]
\[
N(x) = 1 - x
\]

form a De Morgan triplet
Lukasiewicz t-norm, t-conorm, standard negation

\[(x + y - 1) \lor 0\]

\[(x + y) \land 1\]

\[N(x) = 1 - x\]

\[N(T(N(x), N(y))) = S(x, y) \leftrightarrow (x + y) \land 1 = ((1 - x) + (1 - y) - 1) \lor 0 \leftrightarrow min \{x + y, 1\} = max \{1 - x - y, 0\}\]

Case 1: \(1 - x - y < 0 \implies 1 < x + y\)

Case 2: \(1 - x - y > 0 \implies 1 > x + y\)
A. Samuel and M. Balamurugan (2012)
Fuzzy Max-Min Composition Technique in Medical Diagnosis
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The End