An Introduction to Fractal Analysis

Julian Ivaldi

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σ-Algebras and Measures

A measure on a set is a notion of area or weight of certain subsets of that set. These subsets must be a part of a σ-algebra, which is the structure required to define measures. There exist weaker requirements such as algebras or semi-algebras that alone do not suffice to define measures but are still useful.

A measure $\mu$ on a σ-algebra $\mathcal{A}$ is defined as a function $\mu : \mathcal{A} \to [0, \infty]$ that satisfies

1. $\mu(\emptyset) = 0$
2. For $A \subset B$, $\mu(A) \leq \mu(B)$ (Monotonicity)
3. For a countable collection of sets $\{A_i\}$ in $\mathcal{A}$, $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ (Additivity)
Lebesgue Measure, Hausdorff Measure, and More

\[ \mathcal{L}^* (A) = \inf \{ \sum_i |P_i|, A \subset \bigcup_i P_i, P_i \text{ half open rectangles} \} \]

\[ \mathcal{H}_\delta^\alpha (A) = \inf \{ \sum_i |E_i|^\alpha, A \subset \bigcup_i E_i, |E_i| < \delta \} \]

\[ \mathcal{H}^\alpha (A) = \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha (A) \]

\[ \mathcal{H}_\infty^\alpha (A) = \inf \{ \sum_i |E_i|^\alpha, A \subset \bigcup_i E_i \} \]

\[ \mathcal{H}_\infty^\phi (A) = \inf \{ \sum_i \phi(|E_i|), A \subset \bigcup_i E_i \} \]
We define the Hausdorff dimension of a set $A$ to be

$$\dim(A) = \sup\{\alpha : \mathcal{H}^\alpha(A) = \infty\} = \inf\{\alpha : \mathcal{H}^\alpha(A) = 0\}$$

Other ways to define dimension include Minkowski dimension and packing dimension. These other dimensions may bound Hausdorff dimension.
Hutchinson’s Theorem

Hutchinson’s theorem states that given a complete metric space \((X, d)\) and a family of contractions \(\{f_i\}_{i=1}^\ell\) on \(X\),

1. There exists a unique non-empty compact set \(K\) such that

\[
K = \bigcup_{i=1}^\ell f_i(K)
\]

2. For any probability vector \(p = (p_1, ..., p_\ell)\) there exists a unique probability measure \(\mu_p\) on the attractor \(K\) such that

\[
\mu_p = \sum_{i=1}^\ell p_i \mu_p f_i^{-1}
\]

If \(p_i > 0\) for all \(i\), then \(\text{supp}(\mu_p) = K\).
The Banach fixed-point theorem states that for a complete metric space $X$ and a contraction $f$, there exists a unique fixed point $z \in X$ such that $f(z) = z$.

We can make use of this theorem by constructing an appropriate metric space and contraction which prove the existence and uniqueness of a fixed point in that space with desired properties.
For the first statement of Hutchinson’s theorem, we consider the space $\text{Cpt}(X)$ defined as all compact subsets of $X$. Since $X$ is complete, $\text{Cpt}(X)$ endowed with the Hausdorff metric $d_H$ is a complete metric space (Blaschke’s selection theorem).

It can be shown that the function $F = \bigcup_{i=1}^{\ell} f_i$ is a contraction on $(\text{Cpt}(X), d_H)$ and as such, we can apply the fixed-point theorem to obtain a point $K \in \text{Cpt}(X)$ such that $K = F(K) = \bigcup_{i=1}^{\ell} f_i(K)$. 

Proof of Second Claim: Metric Space and Self-Map

We follow a similar process to prove the second claim, this time using the space $P(K)$ of Borel probability measures on $K$. This is a compact metric space when given the dual Lipschitz metric,

$$L(\mu, \nu) = \sup_{\text{Lip}(g) \leq 1} \left| \int gd\mu - \int gd\nu \right|$$

We define a self-map $F_p$ on $P(K)$ as follows

$$F_p(\nu) = \sum_{i=1}^{\ell} p_i \nu f_i^{-1}$$

It remains to show that this is a contraction on $(P(K), L(\mu, \nu))$ and to prove the final note about $\text{supp}(\mu)$. 
Proof of Second Claim: $F_p$ is a Contraction

We first note that for some function $g : K \to \mathbb{R}$ with $\text{Lip}(g) \leq 1$, the Lipschitz norm $\text{Lip}(\sum_{i=1}^{\ell} p_i g f_i) \leq r_{\text{max}}$. Now,

$$L(F_p(\mu), F_p(\nu)) = \sup_{\text{Lip}(g) \leq 1} \left| \int gdF_p(\mu) - \int gdF_p(\nu) \right|$$

$$\left| \int gdF_p(\mu) - \int gdF_p(\nu) \right| = \left| \int \sum_{i=1}^{\ell} p_i g f_i d\mu - \int \sum_{i=1}^{\ell} p_i g f_i d\nu \right|$$

$$\leq \text{Lip} \left( \sum_{i=1}^{\ell} p_i g f_i \right) L(\mu, \nu) \leq r_{\text{max}} L(\mu, \nu)$$

Thus, we see that $F_p$ is a contraction. We can apply the fixed point theorem to obtain $\mu_p$ that satisfies $\mu_p = \sum_{i=1}^{\ell} p_i \mu_p f_i^{-1}$. 

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Support of $\mu_p$ is $K$

If $p_i > 0$ for all $i$, then given a probability measure $\nu \in P(X)$ with bounded support that satisfies $\nu = \sum_{i=1}^{\ell} p_i \nu f_i^{-1}$, we see that $\text{supp}(\nu)$ will satisfy

$$\text{supp}(\nu) = \bigcup_{i=1}^{\ell} f_i(\text{supp}(\nu))$$

Since the support of a measure is always closed, we find that $\text{supp}(\mu_p) = K$ from the uniqueness of $K$. 
The Mass Distribution Principle states that if a set $E$ supports a Borel measure $\mu$ where

$$\mu(B(x, r)) \leq Cr^\alpha$$

for all balls $B(x, r)$ and some constant $0 < C < \infty$, then $\mathcal{H}^\alpha(E) \geq \frac{1}{C} \mu(E)$ and thus $\dim(E) \geq \alpha$. 
Proof of MDP

Consider any cover \( \{U_i\} \) of \( E \). We choose \( \{r_i\} \) such that \( r_i > |U_i| \) and \( \{x_i\} \) where \( x_i \in U_i \). Then, we recall the assumption to state,

\[
\mu(U_i) \leq \mu(B(x_i, r_i)) \leq C r_i^\alpha
\]

We let \( r_i \) approach \( |U_i| \) to conclude \( \mu(U_i) \leq C |U_i|^\alpha \)

\[
\frac{1}{C} \mu(E) \leq \sum_i \frac{\mu(U_i)}{C} \leq \sum_i |U_i|^\alpha
\]

Since \( \{U_i\} \) was arbitrary, we conclude \( \mathcal{H}^\alpha(E) \geq \mathcal{H}_\infty^\alpha(E) \geq \frac{1}{C} \mu(E) \) and thus that \( \dim(E) \geq \alpha \)
Frostman’s Lemma

Frostman’s Lemma states that for a gauge function $\phi$ and a compact set $K \subset \mathbb{R}^d$ with Hausdorff content $\mathcal{H}_\infty^\phi(K) > 0$, there exists a Borel measure $\mu$ on $K$ such that $\mu(K) \geq \mathcal{H}_\infty^\phi(K)$ and for all balls $B$, $\mu(B) \leq C_d \phi(|B|)$
Trees, Flow, and Conductance

A **rooted tree** $\Gamma$ is a collection of vertices and edges starting at a specific root vertex where there exists exactly one path through edges between any two vertices.

We denote the root vertex $\sigma_0$ and for a vertex $\sigma$, the depth from the root $|\sigma|$, and the adjacent vertex closer to the root $\sigma'$.

To each edge $\sigma'\sigma$, we assign a positive **conductance** $C(\sigma'\sigma)$.

We define a **flow** as a non-negative function $f$ of edges such that

$$f(\sigma'\sigma) = \sum_{\tau' = \sigma} f(\sigma\tau)$$

A **legal flow** is one where $f(\sigma'\sigma) \leq C(\sigma'\sigma)$ for all $\sigma$.

The norm of a flow is defined as

$$||f|| = \sum_{|\sigma| = 1} f(\sigma_0\sigma)$$
Example of a Tree
A cut-set is a set of edges $\Pi$ that intersects all paths from the root. A minimal cut-set is one which has no proper subsets that are also cut-sets. Cut-sets have important properties. If we consider a flow $f$, then

$$\|f\| \leq \sum_{e \in \Pi} f(e)$$

Equality holds when $\Pi$ is a minimal cut-set. For legal flows,

$$\|f\| \leq \sum_{e \in \Pi} f(e) \leq \sum_{e \in \Pi} C(e) := C(\Pi)$$
Max-Flow Min-Cut Theorem

The previous slide implies,

\[ \max_{\text{legal flows}} \| f \| \leq \min_{\text{cut sets}} C(\Pi) \]

The max-flow min-cut theorem claims that equality holds for both finite and infinite trees, and most importantly that there exists a flow that attains said maximum value.
To apply our knowledge of trees, we must construct an appropriate tree based on our assumptions.
Fix some integer $b > 1$ and construct the $b$-adic tree $\Gamma$ corresponding to $K$. Vertices of depth $n$ correspond to $b$-adic cubes of generation $n$ that intersect $K$. Thus, all vertices are guaranteed to have a parent. We define conductance on $\Gamma$ as

$$C(\sigma' \sigma) = \phi(\sqrt{db^{-n}})$$

The max-flow min-cut theorem guarantees a maximal flow $f$ for this conductance.
We first construct a premeasure $\tilde{\mu}$ defined as such,

$$\tilde{\mu}(\{\text{all paths through } \sigma'\sigma\}) = f(\sigma'\sigma)$$

If we let $S$ denote the collection of all sets of the form
$$\{\text{all paths through } \sigma'\sigma\} \text{ and } \emptyset,$$
then $S$ is a semi-algebra. $\tilde{\mu}$ is additive by the conservation of flow, so it is a premeasure on $S$. By the extension theorem for semi-algebras, we can extend $\tilde{\mu}$ to the $\sigma$-algebra generated by $S$. Thus, we have constructed a measure $\mu$ which satisfies that $\mu(I_\sigma) = f(\sigma'\sigma)$. It remains to show that it has the desired properties.
\[ \mu(B) \leq C_d \phi(|B|) \]

This property follows from the fact that any cube \( J \) can be covered by \( C_d \) smaller \( b \)-adic cubes, and by the increasing properties of \( \phi \),

\[
\mu(J) \leq \sum_{i=1}^{C_d} \mu(I_\sigma) \leq C_d \phi(|J|)
\]
\[ \mu(K) \geq \mathcal{H}_\infty^\phi(K) \]

First, we recall the tree \( \Gamma \) corresponding to \( K \). We note that any \( b \)-adic cover of \( K \) corresponds to a cut-set of \( \Gamma \), so

\[
\inf_{\Pi} C(\Pi) = \inf_{\Pi} \sum_{e \in \Pi} \phi(\sqrt{db^{-|e|}}) \geq \mathcal{H}_\infty^\phi(K) \geq \mathcal{H}_\infty^\phi(K)
\]

Thus, applying the max-flow min-cut theorem,

\[
\mu(K) = \|f\| = \inf_{\Pi} C(\Pi) \geq \mathcal{H}_\infty^\phi(K)
\]
Sources

Bishop, Peres, Fractals in Probability and Analysis

Chousionis, Measure Theory

Aldridge, Lecture 6, Constructing measures III: Caratheodory’s extension theorem