Metric Space Topology

Tom McGrath

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Outline

- Metric spaces
- Sequences
- Open and closed sets
- Completeness, Compactness, Connectedness
- Theorems from calculus
A metric space is a pair \((M, d)\) where \(M\) is a set of points and \(d\) is a metric that satisfies the following

- **Positive definiteness:** \(d(x, y) \geq 0\). Additionally \(d(x, y) = 0\) if and only if \(x = y\)
- **Symmetry:** \(d(x, y) = d(y, x)\)
- **Triangle inequality:** \(d(x, z) \leq d(x, y) + d(y, z)\)
Examples of Metrics

- Consider $\mathbb{R}$ and the metric $d(x, y)$ for $x, y \in \mathbb{R}$,
  \[ d(x, y) = |x - y| \]

- The Discrete Metric
  \[ d(x, y) = \begin{cases} 
  1 & x \neq y \\
  0 & x = y 
  \end{cases} \]
Definition: Consider a metric space $M$ with the metric $d$. A sequence of points, $a_1, a_2, a_3, \ldots \in M$ denoted $(a_n)$ is a Cauchy sequence if for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, k \in \mathbb{N}$ where $n, k \geq N$,

$$d(a_n, a_k) < \epsilon$$
Convergent Sequences

Definition: Again consider a metric space $M$ with a metric $d$. A sequence $(p_n)$ converges to the limit $p \in M$ if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $n \geq N$,

$$d(p_n, p) < \epsilon$$

Every convergent sequence is a Cauchy sequence, because as the elements of a sequence converge to some point $b$, they must become closer and closer to one another.
Definition: A function $f : M \rightarrow N$ is continuous if it sends convergent sequences in $M$ to convergent sequences in $N$. That is, if $(p_n)$ converges to a limit $p \in M$, then the sequence $(f(p_n))$ converges to the limit $f(p) \in N$. 
Closed Sets and Open Sets

Consider a metric space $M$ with metric $d$. Now consider $S$, a subset of $M$. A point $p \in M$ is a limit of $S$ if there is a sequence of points $(p_n)$ in $S$ such that $(p_n)$ converges to $p$.

- A set is closed if it contains all of its limits
- A set is open if for each $x \in S$ there exists an $r > 0$ such that for $y \in M$, if $d(x, y) < r$, then $y \in S$
- The complement of a closed set is an open set, and the complement of an open set is a closed set
- The topology of a metric space $M$ is the collection $\mathcal{T}$ of all open subsets of $M$
Clopen Sets

Definition: A set is clopen if it is both closed and open

- Consider a metric space $M$, $\emptyset \subset M$. $\emptyset$ is closed since there are no sequences in $\emptyset$, and therefore no limits of sequences in $\emptyset$ that fall outside of it. As well, $\emptyset$ is open since there are no elements in the set, and thus no elements that contradict the condition for the set to be open.

- The complement of $\emptyset$, which is $M$, must then be both open and closed as well.

- Therefore, $M$ and the empty set are clopen sets
Definition: The following are equivalent conditions for continuity of a function $f : M \rightarrow N$, 

- The closed set condition: The preimage of each closed set in $N$ is a closed set in $M$.
- The open set condition: The preimage of each open set in $N$ is an open set in $M$. 
Important Notions of a Metric Space

- Completeness
- Compactness
- Connectedness
Completeness

Definition: A metric space $M$ is complete if each Cauchy sequence in $M$ converges to a limit in $M$.
Definition: A subset $A$ of a metric space $M$ is compact if every sequence $(a_n) \in A$ has a subsequence $(a_{n_k})$ that converges to a limit in $A$. 

Compactness
Theorem: Every compact set is bounded

Proof: Consider a compact subset $A$ of a metric space $M$. Suppose $A$ is not bounded, then for a sequence $(a_n) \subset A$, for any $p \in M$, we have that $d(a_n, p) \to \infty$ as $n$ goes toward infinity. Since $A$ is compact we know that there is some $(a_{n_k}) \subset (a_n)$ such that $a_{n_k} \to p_0 \in M$. This contradicts the fact that $d(a_n, p) \to \infty$, meaning the set must be bounded.
Compactness

Theorem: The closed interval \([a, b] \in \mathbb{R}\) is compact

Proof: Consider a sequence \((x_n)\) in \([a, b]\). We can define the set \(C\) as,

\[
C = \{x \in [a, b] : x_n < x \text{ finitely many times}\}
\]

We can say that \(a \in C\) since there can be no value in the sequence \((x_n)\) that is less than \(a\). \(C\) is not empty. As well, \(b\) is an upper bound for \(C\) since there are no \(x \in [a, b]\) that are greater than \(b\). There must exist some least upper bound of \(C\), \(c \in [a, b]\). Suppose there is no subsequence of \((x_n)\) that converges to \(c\). Then for some \(r > 0\), \(x_n < c + r\) finitely many times, since by our assumption the sequence does not converge to \(c\). Thus, \(c + r \in C\), a contradiction to the fact that \(c\) is the least upper bound of \(C\). Therefore, there must be a subsequence of \((x_n)\) that converges to \(c\) and it follows that \([a, b]\) is compact.
Compactness

Theorem: The Cartesian product of two compact sets is compact.

Proof: Consider metric spaces $M$ and $N$, where $A \subset M$, $B \subset N$. Suppose $A$ and $B$ are compact. We can define a sequence $(a_n, b_n)$ in $A \times B$. Since $A$ is compact, $(a_n)$ has a subsequence $(a_{n_k})$ that converges to a point $a \in A$. As well, since $B$ is compact, the sequence $(b_{n_k})$ has a subsequence $(b_{n_{k_l}})$ that converges to a point $b \in B$. It follows that, $(a_{n_{k_l}}, b_{n_{k_l}})$ converges to $(a, b) \in A \times B$. Thus, the Cartesian product is compact.

Suppose the Cartesian product of $n$ compact sets is compact. Then the Cartesian product of $n + 1$ compact sets,

$$[A_1 \times A_2 \times ... \times A_n] \times A_{n+1}$$

This is the Cartesian product of two compact sets which we know to be compact. By induction the Cartesian product of $m \in \mathbb{N}$ compact sets is compact.
Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

Proof: Every bounded sequence in $\mathbb{R}^m$ can be contained in a box, this box being the Cartesian product of intervals, with $a_i, b_i \in \mathbb{R}$,

$$[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_m, b_m]$$

We have shown that each $[a, b]$ is compact, and that the Cartesian product of compact sets is compact, and it follows that a box in $\mathbb{R}^m$ is compact. Thus any sequence in this box must have a convergent subsequence.
Compactness

Theorem: If $f : M \rightarrow N$ is continuous and $A$ is a compact subset of $M$, then $f(A)$ is a compact subset of $N$.

Proof: Suppose $(b_n)$ is a sequence in $f(A) = \{ f(x) : x \in A \}$. We can assign each $b_n \in f(A)$ with an $a_n \in A$ such that $f(a_n) = b_n$. Since $A$ is compact, there is a subsequence $(a_{n_k})$ that converges to a point $p \in A$. It follows that $f(a_{n_k}) = b_{n_k}$ which converges to $f(p) \in f(A)$. Therefore, for each sequence $(b_n)$ in $f(A)$, there is a subsequence $(b_{n_k})$ that converges to a point $f(p) \in f(A)$. Thus $f(A)$ is compact.
Connectedness

Definition: Given a metric space $M$, if $M$ has a proper clopen subset $A$, that is $A$ is neither $M$ nor $\emptyset$, then $M$ is disconnected. $M$ is connected if it is not disconnected, that is, there are no proper clopen subsets of $M$.

If there are proper clopen subsets of a metric space, $A$ and $A^c$, then we can separate $M$ into nonempty disjoint sets,

$$M = A \cup A^c$$
Connectedness

Theorem: If $M$ is connected and the function $f : M \rightarrow N$ is continuous and surjective, then $N$ is connected.

Proof: Suppose $A$ is a proper clopen subset $N$. Let $X = \{ m \in M : f(m) \in A \}$. $X$ is the preimage of $A$. This preimage $X$ must be clopen since $f$ is continuous. It must be nonempty since $f$ is surjective. It follows that the preimage of $A^c$ must be nonempty as well, implying that $X$ is neither empty nor the set $M$. Thus $X$ is a proper clopen subset of $M$ contradicting the fact that $M$ is connected. It must be that a proper clopen subset of $N$ cannot exist, and thus $N$ is connected.
Connectedness

Theorem: \( \mathbb{R} \) is connected.

Proof: Suppose we have some nonempty clopen subset \( U \subset \mathbb{R} \). If we take some \( p \in U \), we can make a set,

\[
X = \{ x \in U : \text{the open interval} \ (p, x) \subset U \}
\]

\( X \) is nonempty since \( U \) is open. Let \( s \) be the supremum of \( X \). If \( s \) is finite, \( s \) must be the least upper bound of \( X \) and \( s \) is a limit of \( U \) since \( s \) is the greatest value such that \((p, s) \subset U\). Since \( U \) is closed, \( s \in U \). Since \( U \) is open, for some \( r > 0 \), we know that the interval \((s - r, s + r) \subset U \). Thus, \( s + r \in X \), contradicting the fact that \( s \) is the least upper bound of \( X \). Therefore it must be that \( X \) is unbounded above and the interval \((p, \infty) \subset U \).

Repeating this process with the greatest lower bound gives the result that \((-\infty, p) \subset U \). Thus, \( U = \mathbb{R} \), there are no proper clopen subsets of \( \mathbb{R} \), and \( \mathbb{R} \) is connected.
Theorems from Calculus

- Intermediate Value Theorem
- Extreme Value Theorem (Minimum-Maximum)
Intermediate Value Theorem: A continuous function defined on an interval \([a, b]\) achieves all intermediate values. If \(f(a) = \alpha\), \(f(b) = \beta\), and \(\gamma\) is given such that \(\alpha \leq \gamma \leq \beta\), then there is some \(c \in [a, b]\) such that \(f(c) = \gamma\).

We can first prove the general intermediate value theorem.
Intermediate Value Theorem

General Intermediate Value Theorem: Every continuous real valued function defined on a connected domain attains all intermediate values.

Proof: Consider $M$, a connected metric space, and a function $f : M \rightarrow \mathbb{R}$ which is continuous. As well, $f(a) = \alpha$ and $f(b) = \beta$, where $\alpha < \beta$. Now suppose there exists some $\gamma$, $\alpha < \gamma < \beta$, such that there is no $x \in M$ where $f(x) = \gamma$. Then we can split up $M$ into two open disjoint sets,

$$M = \{x \in M : f(x) < \gamma\} \cup \{x \in M : f(x) > \gamma\}$$

This representation contradicts the fact that $M$ is connected, and therefore $f(x)$ must attain $\gamma$. That is, the function attains all intermediate values.
Intermediate Value Theorem

Theorem: The interval \([a, b]\) is connected.

Proof: Define a function \(f : \mathbb{R} \rightarrow [a, b]\) as follows,

\[
f(x) = \begin{cases} 
  a, & x \leq a \\
  x, & a < x < b \\
  b, & x \geq b
\end{cases}
\]

This is a surjective continuous function from \(\mathbb{R}\) to a closed interval \([a, b] \subset \mathbb{R}\). We have shown that \(\mathbb{R}\) is connected and from an earlier theorem, this implies that \([a, b]\) is also connected.
We can now prove the Intermediate Value Theorem that we initially presented.

Proof: Apply the General Intermediate Value Theorem to the connected domain \([a, b]\).
Extreme Value Theorem: A continuous function $f$ defined on an interval $[a, b]$ takes on absolute minimum and absolute maximum values, that is for some $x_0, x_1 \in [a, b]$ and for all $x \in [a, b]$, 
$f(x_0) \leq f(x) \leq f(x_1)$. 
Extreme Value Theorem

We can use some previous theorems to prove the Extreme Value Theorem.

Proof: We have shown that \([a, b]\) is compact, compact sets are bounded, and that the continuous image of a compact set is compact. Therefore a continuous function \(f\) defined on \([a, b]\) has bounds \(m, M\) such that for all \(x \in [a, b]\), \(m \leq f(x) \leq M\). Let \(M, m\) be the supremum and infimum of the set \(\{f(x) : x \in [a, b]\}\) respectively. Thus, there is a sequence \((x_n) \in [a, b]\) such that \(f(x_n) \to M\). By compactness, there exists a subsequence \((x_{n_k}) \to x_1 \in [a, b]\) and \(f(x_{n_k}) \to f(x_1)\). We also have that \(f(x_{n_k}) \to M\). Therefore, \(f(x_1) = M\). By symmetry, \(f(x_0) = m\) for some \(x_0 \in [a, b]\). Thus, we have for all \(x \in [a, b]\), \(f(x_0) \leq f(x) \leq f(x_1)\).
The following is the textbook that we used throughout the program, and where I referenced to complete the proofs in this presentation.

Thank you for listening to my presentation, and thank you to my mentor Geoff Lindsell and the organizers of the Directed Reading Program