Notation and conventions:

- Denote by $\mathbb{C}$ the complex plane and $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is the open unit disk.
- A region means a nonempty connected open set.
- The terminology analytic function and holomorphic function may be used interchangeably.

Problem 1. Let $f$ be a nonconstant smooth function on $\mathbb{C}$ such that the set $\Gamma$ given by $\Gamma = \{ z \in \mathbb{C} : |f(z)| = 7 \}$ is a smooth simple closed curve in $\mathbb{C}$. Denote by $G$ the bounded region enclosed by $\Gamma$. Assume $f$ is holomorphic in $G$. Prove that $f$ has at least one zero in $G$.

Problem 2. Let $g$ be an entire function satisfying
\[ \max_{|z| \leq R} |g(z)| \leq R^9, \text{ for all } R \geq 200. \]
Show that $g$ is a polynomial of degree at most 9.

Problem 3. How many zeros counting multiplicities does the function
\[ \psi(z) = z^8 - 6e^z + 5 \]
have in the region $\{ z \in \mathbb{C} : |z| < 2 \}$? Prove your assertion.

Problem 4. Let $U = \{ re^{i\theta} : 0 < r < 2, -\pi < \theta < \pi/2 \}$. Explicitly describe a one-to-one conformal map from $U$ onto the unit disk $\mathbb{D}$.

Problem 5. Let $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$. For all holomorphic functions $h$ in $\mathbb{H}$ such that $h(i) = 0$ and $|h(z)| < 1$ for all $z \in \mathbb{H}$, find the largest possible value of $|h(6i)|$.

Problem 6. Let $\mathcal{C} = \{ z \in \mathbb{C} : |z| = 10^5 \}$ with the positive direction. Evaluate the integral
\[ \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{z^{2020}}{\prod_{k=1}^{2021}(z-k)} \, dz. \]

Problem 7. Let $f$, $\Gamma$, and $G$ be given as in Problem 1. Assume in addition that $\Gamma$ contains no zero of $f' \equiv \partial f / \partial z$. Suppose $f$ has $m$ zeros counting multiplicities in $G$. How many zeros counting multiplicities does $f'$ have in $G$? Prove your assertion.