# COMPLEX ANALYSIS PRELIM

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**Notation and conventions**:
- Denote by $\mathbb{C}$ the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.
- A *region* means a nonempty connected open set.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

**Problem 1.** Let $f$ be a nonconstant smooth function on $\mathbb{C}$ such that the set $\Gamma$ given by $\Gamma = \{z \in \mathbb{C} : |f(z)| = 7\}$ is a smooth simple closed curve in $\mathbb{C}$. Denote by $G$ the bounded region enclosed by $\Gamma$. Assume $f$ is holomorphic in $G$. Prove that $f$ has at least one zero in $G$.

**Problem 2.** Let $g$ be an entire function satisfying

$$\max_{\{|z| \leq R\}} |g(z)| \leq R^9, \quad \text{for all } R \geq 200.$$

Show that $g$ is a polynomial of degree at most 9.

**Problem 3.** How many zeros counting multiplicities does the function

$$\psi(z) = z^8 - 6e^z + 5$$

have in the region $\{z \in \mathbb{C} : |z| < 2\}$? Prove your assertion.

**Problem 4.** Let $U = \{re^{i\theta} : 0 < r < 2, -\pi < \theta < \pi/2\}$. Explicitly describe a one-to-one conformal map from $U$ onto the unit disk $\mathbb{D}$.

**Problem 5.** Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. For all holomorphic functions $h$ in $\mathbb{H}$ such that $h(i) = 0$ and $|h(z)| < 1$ for all $z \in \mathbb{H}$, find the largest possible value of $|h(6i)|$.

**Problem 6.** Let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 10^5\}$ with the positive direction. Evaluate the integral

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{z^{2020}}{\prod_{k=1}^{2021} (z - k)} \, dz.$$

**Problem 7.** Let $f$, $\Gamma$, and $G$ be given as in Problem 1. Assume in addition that $\Gamma$ contains no zero of $f' \equiv \partial f / \partial z$. Suppose $f$ has $m$ zeros counting multiplicities in $G$. How many zeros counting multiplicities does $f'$ have in $G$? Prove your assertion.