Topology Prelim, January 2021

1. Suppose X and Y are topological spaces, and $f : X \to Y$ is a map satisfying $\mathrm{Int} f^{-1}(B) \subseteq f^{-1}(\mathrm{Int} B)$ for all $B \subseteq Y$. Is f an open map? Prove your assertion.

2. Let $L_n \subset \mathbb{R}^2$ denote the closed line segment joining the point $(0,0)$ to the point $(\frac{1}{n}, 1)$. Consider $X = \bigcup_{n=1}^{\infty} L_n \cup \{(0,1)\}$ with its subspace topology induced from $\mathbb{R}^2$.
(1). Is X connected? Prove your assertion.
(2). Is X path-connected? Prove your assertion.

3. Let X be a locally compact Hausdorff space. Let ∞ be some object not in X and consider $X^* = X \sqcup \{\infty\}$ with the following topology:

$$\mathcal{T} = \{\text{open subsets of } X\} \cup \{U \subseteq X^* : X^* \setminus U \text{ is a compact subset of } X\}.$$

(1). Show that X^* is a compact Hausdorff space.
(2). Show that X is dense in X^* if and only if X is noncompact.

4. Let $p : E \to X$ be a covering map with $p(e_0) = x_0$. The lifting correspondence is denoted by

$$\phi : \pi_1(X, x_0) \to p^{-1}(x_0).$$

Show that if E is simply connected, then ϕ is bijective.

5. Let X be a path-connected topological space. Show that every map $\mathbb{S}^1 \to X$ is homotopic to a constant map iff $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

6. Let X be the space obtained from $\mathbb{R}^3$ by removing the x-axis, the straight line $C_1 = \{(x, 2, 2) : x \in \mathbb{R}\}$, and the circle $C_2 = \{(0, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 4\}$. Compute $\pi_1(X)$.