MATH 5510 NUMERICAL ANALYSIS   January 2021   PRELIMINARY EXAMINATION

INSTRUCTIONS: Solve three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) Solve 4 problems out of (a), (b), (c), (d). (e)

(a) Consider the following classical interpolation problem.

- **Given** $n+1$ support points

$$(x_i, f_i) \qquad i = 0, \dots, n; \qquad\qquad (x_i \neq x_j \quad for \quad i \neq j).$$

- **Find** a polynomial $P(x)$ whose degree does not exceed n such that

$$P(x_i) = f_i, \qquad i = 0, \dots n.$$

Define the Vandermonde matrix, and then reformulate the above interpolation problem as a matrix problem of solving a linear system of equations with the Vandermonde coefficient matrix.
Use the condition

$$x_i \neq x_j \quad for \quad i \neq j,$$

to prove that the Vandermonde matrix is nonsingular.
Use the latter fact to prove that the classical interpolation problem stated above has a unique solution.

(b) Let $P_{i_0 i_1 \dots i_k}(x)$ be the (unique) polynomial that interpolates at points

$$(x_{i_m}, f_{i_m}) \qquad m = 0, \dots k.$$

Prove the Neville formula

$$P_{0,1,2,\dots,k}(x) = \frac{(x - x_j)P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x - x_i)P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{x_i - x_j}$$

(c) Prove that there exists a unique coefficient $f_{i_0 \dots i_k}$ such that

$$P_{i_0 \dots i_k}(x) = P_{i_0 \dots i_{k-1}} + f_{i_0 \dots i_k}(x - x_{i_0})(x - x_{i_1}) \cdots (x - x_{i_{k-1}}).$$

(d) Prove the recursion:

$$f_{i_0 \dots i_k} = \frac{f_{i_1 \dots i_k} - f_{i_0 \dots i_{k-1}}}{x_{i_k} - x_{i_0}}.$$

(e) Prove the following theorem (error in polynomial interpolation).

If the function f has an $(n+1)$st derivative, then for every argument $\bar{x}$ there exist a number ξ (in the smallest interval containing $x_{i_0}, x_{i_1}, \dots x_{i_n}, \bar{x}$), satisfying

$$f(\bar{x}) - P_{i_0,i_1,\dots,i_n}(x) = \frac{w(\bar{x}) f^{(n+1)}(\xi)}{(n+1)!},$$

where

$$w(x) = (x - x_{i_0})(x - x_{i_1}) \dots (x - x_{i_n}).$$

Q2) Solve (a), (b), (c)

(a) Use the fact that each norm $\|\cdot\|$ on $\mathbb{C}^n$ is uniformly continuous (no need to prove the latter fact, just formulate it as a specific inequality) to prove the following theorem.

All norms on $\mathbb{C}^n$ are equivalent in the following sense. For each pair of norms $p_1(x)$ and $p_2(x)$ there are positive constants m and M satisfying

$$mp_2(x) \le p_1(x) \le Mp_2(x)$$

for all x.

(b) Prove that if F is an $n \times n$ matrix with $\|F\| < 1$, then $(I+F)^{-1}$ exists and satisfies

$$\|(I+F)^{-1}\| \le \frac{1}{1 - \|F\|}.$$

(c) Let A be a nonsingular $n \times n$ matrix, $B = A(I+F)$, $\|F\| < 1$, and x and Δx be defined by

$$Ax = b, \qquad B(x + \Delta x) = b.$$

Use **(b)** to prove that

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{\|F\|}{1 - \|F\|}$$

as well as

$$\frac{\|\Delta x\|}{\|x\|} \le \frac{cond(A)}{1 - cond(A)\frac{\|B-A\|}{\|A\|}} \cdot \frac{\|B-A\|}{\|A\|}$$

if

$$cond(A)\frac{\|B-A\|}{\|A\|} < 1.$$

Q3) Answer 3 out of 4 questions (a), (b), (c), (d).

(a) Define a Hankel matrix. Let H be an $n \times n$ positive definite Hankel matrix. Relate the factorization

$$H\widetilde{U} = \widetilde{L} \tag{1}$$

to the standard LDL^* factorization of H to prove that (1) always exists and it is unique. Here $\widetilde{U}$ is a unit (i.e., with 1's on the main diagonal) upper triangular matrix, and $\widetilde{L}$ is a lower triangular matrix.

(b) Let $\langle \cdot, \cdot \rangle$ be an inner product in the vector space Π_n (of all polynomials whose degree does not exceed n). Let the above Hankel matrix H be a moment matrix, i.e., $H = [\langle x^i, x^j \rangle]_{i,j=0}^n$. Let

$$u_k(x) = u_{0,k} + u_{1,k}x + u_{2,k}x^2 + \ldots + u_{k-1,k}x^{k-1} + x^k. \qquad (2)$$

be the k-th orthogonal polynomial with respect to $\langle \cdot, \cdot \rangle$. Prove that the k-th column of the matrix $\widetilde{U}$ of (a) contains the coefficients of $u_k(x)$ as in

$$\widetilde{U} = \begin{bmatrix} 1 & u_{0,1} & u_{0,2} & u_{0,3} & \cdots & \cdots & u_{0,n} \\ 0 & 1 & u_{1,2} & u_{1,3} & \cdots & \cdots & u_{1,n} \\ 0 & 0 & 1 & u_{2,3} & \cdots & \cdots & u_{2,n} \\ \vdots & & 0 & 1 & \cdots & \cdots & u_{3,n} \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & 1 & u_{n-1,n} \\ 0 & & & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

(c) Derive a algorithm to compute the columns of $\widetilde{U}$ based on the formula (deduce it) that relates the k-th column u_k of U to its two "predecessors" u_{k-2}, u_{k-1} $(k = 3, \ldots, n)$.

(d) Prove that the algorithm of (c) uses $O(n^2)$ arithmetic operations.

Q4) Answer 4 out of 5 questions (a), (b), (c), (d), (e).

Derive a fast $O(n \log n)$ FFT-based algorithm for the polynomial multiplication problem, that is, given coefficients of two polynomials $a(x), b(x)$, compute the coefficients of their product $c(x) = a(x)b(x)$.

(a) Prove that the above polynomial multiplication problem is equivalent to the problem of multiplying a lower triangular Toeplitz matrix by a vector.

(b) Show how to "embed" a Toeplitz matrix into a circulant matrix, and justify the fact that the problem of (a) (that is, of multiplying a lower triangular Toeplitz matrix by a vector) can be solved via multiplying a circulant matrix by a vector.

(c) Prove that any circulant matrix C admits a factorization

$$C = FDF^*$$

where F is the DFT matrix and D is a diagonal matrix.

(d) Deduce the formula for the diagonal entries of D.

(e) Describe "in words" how the results of (a), (b), (c), and (d) allow us to compute the coefficients of $c(x) = a(x)b(x)$ in $O(n \log n)$ arithmetic operations.