Applied Math Prelim August 2021

1. Consider the domain $\mathbb{R}^3$. Define $u(x) = -\frac{1}{4\pi|x|}$ for $x \in \mathbb{R}^3$. Show that u is a fundamental solution of the Laplace operator Δ, i.e. $\Delta u = \delta$ in distributional sense where δ is the Dirac-delta distribution.

2. (2a) Let X be a Hilbert space and $A_n : X \to X$ be a linear bounded compact operator for all $n = 1, 2, \ldots$. Suppose $A_n \to B$ in operator norm for some linear bounded operator $B : X \to X$. Show that B is a compact operator as well.

   (2b) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in the Hilbert space X. Suppose $B : X \to X$ is a linear bounded operator with $\sum_{j=1}^{\infty} \|Be_j\|^2 < \infty$. (This is known as the Hilbert-Schmidt operator). For any $w \in X$, define $A_n : X \to X$ such that $A_n w = \sum_{j=1}^{n} \langle w, e_j \rangle Be_j$. Show that A_n is a linear bounded compact operator and $A_n \to B$ in operator norm.

   (2c) Given $G : (0,1) \times (0,1) \to \mathbb{R}$ satisfying $\int_0^1 \int_0^1 |G(x,y)|^2 \, dx \, dy < \infty$. For any $u \in L^2(0,1)$ define $B : L^2(0,1) \to L^2(0,1)$ such that $Bu\,(x) = \int_0^1 G(x,y)u(y)\,dy$. Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of L^2, use the Parseal's relation or otherwise to show $\int_0^1 \int_0^1 |G(x,y)|^2 \, dx \, dy = \sum_{j=1}^{\infty} \|Be_j\|^2$. i.e. B is a Hilbert-Schmidt operator.

3. (3a) Find the Green's function $G(x, y)$ for the operator A where

$$Au \equiv u'' - u$$

with $u(0) = u(\pi) = 0$.

(3b) Define $T : L^2(0, \pi) \to L^2(0, \pi)$ such that for any $f \in L^2(0, \pi)$,

$$(Tf)(x) = \int_0^\pi G(x, y) f(y)\, dy \,.$$

Show that T is a self-adjoint linear bounded operator. Explain why T is a compact operator (See question (2c)).

(3c) Explain what spectral theorem is and why it is applicable to T.

4. Let $f : D \to Y$ be a mapping from an open set D in a normed linear space X to another normed linear space Y.

(4a) State the definition of Fréchet derivative of f.

(4b) Observe that $W \equiv \{w \in C^1[0, 1] : w(0) = w(1) = 0\}$ is a Banach space and consider $\mathcal{I} : W \to \mathbb{R}$ such that for any $u \in W$

$$\mathcal{I}(u) = \int_0^1 (\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4)\, dx \,.$$

Show that $\mathcal{I}$ is Frechet differentiable at any $u \in W$ and find this derivative.

5. Let X be a normed linear space.

(5a) Define weak convergence and strong convergence in X.

(5b) Show that the weak limit is unique.

(5c) Show that strong convergence implies weak convergence.

(5d) Find a weak convergent sequence which is not strongly convergent.