Probability Prelim Exam for Actuarial Students
August 2021

Instructions

(a). The exam is closed book and closed notes.

(b). Answers must be justified whenever possible in order to earn full credit.

(c). Points will be deducted for mathematically incorrect statements.

1. (10 points) Let Ω be a non-empty set and $\mathscr{F}_1$, $\mathscr{F}_2$, ... be a sequence of σ-algebras on Ω such that $\mathscr{F}_i \subseteq \mathscr{F}_{i+1}$ for all $i \geq 1$. Prove or disprove (with a counterexample) that $\bigcup_{i=1}^{\infty} \mathscr{F}_i$ is a σ-algebra.

2. (10 points) Find random variables X, Y, and Z on a probability space $(\Omega, \mathscr{F}, P)$ such that both the following conditions hold:
   - $P(X > Y)P(Y > Z)P(Z > X) > 0$, and
   - $E[X] = E[Y] = E[Z] = 1$.

3. (10 points) Let $n \geq 2$ be an integer and $U_1, U_2, \ldots, U_n$ be independent and uniformly distributed random variables on $(0, 1)$. Let $w_1, w_2, \ldots, w_n$ be positive real numbers. Calculate
$$P\left(w_1^{U_1} > \max\left\{w_i^{U_i} : i = 2, 3, \ldots, n\right\}\right).$$

4. (10 points) Let $X_1, X_2, \ldots$ be a sequence of random variables satisfying $E[|X_n|] \leq 1$ for all n. Suppose that X is a random variable such that $\lim_{n\to\infty} E[|X_n - X|] = 0$. Let $\epsilon > 0$. Prove or disprove that
$$P(|X_n - X| \geq \epsilon \text{ i.o.}) = 0.$$

5. (10 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Bern(p) (i.e., $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$) random variables for some $p \in (0, 1)$. Let $S = X_1 + \cdots + X_n$. Find $\mathrm{Var}(X_1|S)$, the conditional variance of X_1 with respect to $\sigma(S)$, defined through
$$\mathrm{Var}(X_1|S) = E[(X_1 - E[X_1|S])^2|S].$$

6. (10 points) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Find the value of c such that
$$B_t^3 - ctB_t$$
is a martingale.

7. (10 points) Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Let $n \geq 2$ be an integer. Find
$$E[B_1|B_n > 0].$$