Real Analysis Preliminary Exam, August 2021

Instructions and notation:

(i) Complete all problems. Give full justifications for all answers in the exam booklet.

(ii) Outer Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^{*n}$. Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$.

1. (10 points) Let $(X, \mathcal{A}, \mu)$ be a finite measure space such that $\mu(X) > 0$ and let $f : X \to \mathbb{R}$ be an $\mathcal{A}$-measurable function such that $f(x) \in (a, b)$ for μ-a.e. $x \in X$ where $a, b \in \mathbb{R}, a < b$. Show that
$$a\mu(X) < \int f \, \mathrm{d}\mu < b\mu(X).$$

2. (10 points) Let $f : \mathbb{R}^n \to (0, +\infty)$ be a Lebesgue measurable function with $\|f\|_{L^1(\mathbb{R}^n)} = 1$. Show that if $E \subset \mathbb{R}^n$ is Lebesgue measurable with $\mathcal{L}^n(E) \in (0, \infty)$, then
$$\int_E \log f(x) \, \mathrm{d}\mathcal{L}^n(x) \leq -\mathcal{L}^n(E) \log(\mathcal{L}^n(E)).$$

3. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f, g_n, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable functions such that $f_n \to f, g_n \to g$ in measure. Prove that if for all $n \in \mathbb{N}$,
$$\sup_{x \in X} |f_n(x)| \leq 1 \text{ and } \sup_{x \in X} |g_n(x)| \leq 1,$$
then $f_n g_n \to fg$ in measure.

4. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that there exists some $C \geq 0$ such that
$$|f(x) - f(y)| \leq C|x - y| \text{ for all } x, y \in \mathbb{R}.$$
   1. Prove that $\mathcal{L}^{*1}(f(A)) \leq C\mathcal{L}^{*1}(A)$ for all $A \subset \mathbb{R}$.
   2. Prove that f maps Lebesgue measurable sets to Lebesgue measurable sets.

5. (10 points) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $f : X \to [0, \infty)$ be an $\mathcal{A}$-measurable function. Prove that
$$\int_X f(x) \, \mathrm{d}\mu(x) = \int_{[0,\infty)} \mu(\{x \in X : f(x) > y\}) \, \mathrm{d}\mathcal{L}^1(y).$$

6. (10 points) Let $f, f_i : \mathbb{R}^n \to \mathbb{R}, i \in \mathbb{N}$, be Lebesgue measurable functions and let $g \in L^1(\mathbb{R})$. Show that if $f_i(x) \geq 0$ for all $i \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $f_i \to f$ pointwise and
$$\int f_i \, \mathrm{d}\mathcal{L}^n \leq \int_i^{2i} g \, \mathrm{d}\mathcal{L}^1 \text{ for all } i \in \mathbb{N},$$
then $f(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

7. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n, f \in L^1(\mu), n \in \mathbb{N}$, such that $f_n \to f$ in $L^1(\mu)$. Show that if $\sup_{n \in \mathbb{N}} \|f_n\|_{L^4(\mu)} < \infty$ then $f \in L^2(\mu)$ and $f_n \to f$ in $L^2(\mu)$.