1. (10 pts) In the group $G = \text{GL}_2(\mathbb{C})$, let $S = \text{SL}_2(\mathbb{C}) = \{A \in G : \det A = 1\}$ and $N = \mathbb{C}^x I_2 := \{ (z \ 0) : z \in \mathbb{C}^x \}$. Both $S$ and $N$ are subgroups of $G$ (no need to show that).
   
   (a) (3 pts) Show $N$ is a normal subgroup of $G$.
   
   (b) (3 pts) Show $SN = G$ and $S \cap N = \{ \pm I_2 \}$.
   
   (c) (4 pts) Using (b), prove the quotient groups $\text{GL}_2(\mathbb{C})/(\mathbb{C}^x I_2)$ and $\text{SL}_2(\mathbb{C})/\{ \pm I_2 \}$ are isomorphic.

2. (10 pts)
   
   (a) (5 pts) Let $p < q$ be primes such that $q \not\equiv 1 \mod p$, and let $G$ be a group of order $pq$. Prove that $G$ is cyclic.
   
   (b) (5 pts) Use semi-direct products to give an example of a non-abelian group of order 21. After describing the group, (i) show it has order 21 and (ii) show two explicit elements do not commute.

3. (10 pts)
   
   (a) (5 pts) Let $R$ be a UFD and $P(X) = X^3 + a_2X^2 + a_1X + a_0 \in R[X]$. Prove that $P(X)$ is irreducible in $R[X]$ if and only if $P(X)$ does not have a root in $R$.
   
   (b) (5 pts) Prove that $X^4 + X^2y^2 + Y^3$ is irreducible in $\mathbb{C}[X,Y]$.

4. (10 pts)
   
   (a) (4 pts) Let $R$ be a commutative ring. Suppose that $I = (a_1, a_2, \ldots, a_m)$ and $J = (b_1, b_2, \ldots, b_n)$ are ideals in $R$. Show that the product ideal $IJ$ is the ideal generated by all products $a_ib_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
   
   (b) (6 pts) Let $R = \mathbb{Z}[\sqrt{-5}]$, and consider the ideals $I = (2, 1 + \sqrt{-5})$ and $J = (3, 2 + \sqrt{-5})$. Prove that $IJ$ is principal by determining a generator.

5. (10 pts) In this problem, $A$ is an $n \times n$ real symmetric matrix, where $n \geq 1$.
   
   (a) (4 pts) State (but don’t prove) the spectral theorem for the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ and for the dot product (standard inner product) on $\mathbb{R}^n$.
   
   (b) (3 pts) If $A^2 = A$, then show $Av \perp (v - Av)$ for all $v \in \mathbb{R}^n$.
   
   (c) (3 pts) For $v$ and $w$ in $\mathbb{R}^n$, set $\langle v, w \rangle_A = v \cdot Aw$. Show $\langle \cdot, \cdot \rangle_A$ is an inner product on $\mathbb{R}^n$ (i.e., $\langle \cdot, \cdot \rangle_A$ is a positive-definite symmetric bilinear form on $\mathbb{R}^n$) if all the eigenvalues of $A$ are positive. This part is unrelated to part (b).

6. (10 pts) Give examples as requested, with justification.
   
   (a) (2.5 pts) A group isomorphism $f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/15\mathbb{Z})^\times$.
   
   (b) (2.5 pts) An integral domain with an explicit irreducible element that is not prime.
   
   (c) (2.5 pts) The statement of a theorem in algebra whose proof uses Zorn’s lemma.
   
   (d) (2.5 pts) An explicit element of the dual space $(\mathbb{R}^3)^\ast$ that vanishes on the vectors $(2, 1, 0)$ and $(0, 1, 2)$. 