Abstract Algebra Prelim Jan. 2022

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. (**10 pts**) In the group $G = \mathrm{GL}_2(\mathbf{C})$, let $S = \mathrm{SL}_2(\mathbf{C}) = \{A \in G : \det A = 1\}$ and $N = \mathbf{C}^\times I_2 := \{\left(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right) : z \in \mathbf{C}^\times\}$. Both S and N are subgroups of G (no need to show that).
   (a) (**3 pts**) Show N is a normal subgroup of G.
   (b) (**3 pts**) Show $SN = G$ and $S \cap N = \{\pm I_2\}$.
   (c) (**4 pts**) Using (b), prove the quotient groups $\mathrm{GL}_2(\mathbf{C})/(\mathbf{C}^\times I_2)$ and $\mathrm{SL}_2(\mathbf{C})/\{\pm I_2\}$ are isomorphic.

2. (**10 pts**)
   (a) (**5 pts**) Let $p < q$ be primes such that $q \not\equiv 1 \bmod p$, and let G be a group of order pq. Prove that G is cyclic.
   (b) (**5 pts**) Use semi-direct products to give an example of a non-abelian group of order 21. After describing the group, (i) show it has order 21 and (ii) show two explicit elements do not commute.

3. (**10 pts**)
   (a) (**5 pts**) Let R be a UFD and $P(X) = X^3 + a_2X^2 + a_1X + a_0 \in R[X]$. Prove that $P(X)$ is irreducible in $R[X]$ if and only if $P(X)$ does not have a root in R.
   (b) (**5 pts**) Prove that $X^4 + X^2Y^2 + Y^3$ is irreducible in $\mathbf{C}[X, Y]$.

4. (**10 pts**)
   (a) (**4 pts**) Let R be a commutative ring. Suppose that $I = (a_1, a_2, \ldots, a_m)$ and $J = (b_1, b_2, \ldots, b_n)$ are ideals in R. Show that the product ideal IJ is the ideal generated by all products a_ib_j for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.
   (b) (**6 pts**) Let $R = \mathbf{Z}[\sqrt{-5}]$, and consider the ideals $I = (2, 1 + \sqrt{-5})$ and $J = (3, 2 + \sqrt{-5})$. Prove that IJ is principal by determining a generator.

5. (**10 pts**) In this problem, A is an $n \times n$ real symmetric matrix, where $n \geq 1$.
   (a) (**4 pts**) State (but don't prove) the spectral theorem for the linear mapping $A\colon \mathbf{R}^n \to \mathbf{R}^n$ and for the dot product (standard inner product) on $\mathbf{R}^n$.
   (b) (**3 pts**) If $A^2 = A$, then show $A\mathbf{v} \perp (\mathbf{v} - A\mathbf{v})$ for all $\mathbf{v} \in \mathbf{R}^n$.
   (c) (**3 pts**) For $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$, set $\langle \mathbf{v}, \mathbf{w} \rangle_A = \mathbf{v} \cdot A\mathbf{w}$. Show $\langle \cdot, \cdot \rangle_A$ is an inner product on $\mathbf{R}^n$ (*i.e.*, $\langle \cdot, \cdot \rangle_A$ is a positive-definite symmetric bilinear form on $\mathbf{R}^n$) if all the eigenvalues of A are positive. This part is unrelated to part (b).

6. (**10 pts**) Give examples as requested, with justification.
   (a) (**2.5 pts**) A group isomorphism $f\colon \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \to (\mathbf{Z}/15\mathbf{Z})^\times$.
   (b) (**2.5 pts**) An integral domain with an explicit irreducible element that is not prime.
   (c) (**2.5 pts**) The statement of a theorem in algebra whose proof uses Zorn's lemma.
   (d) (**2.5 pts**) An explicit element of the dual space $(\mathbf{R}^3)^*$ that vanishes on the vectors $(2, 1, 0)$ and $(0, 1, 2)$.