COMPLEX ANALYSIS PRELIM

JANUARY 2022

Notation and conventions:

- Denote by $\mathbb{C}$ the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk.
- A *region* means a nonempty connected open set.
- The terminology *analytic* function and *holomorphic* function may be used interchangeably.

Problem 1. Let $F(z) = \dfrac{a - bz}{c - z}$. Find necessary and sufficient conditions on $a, b, c \in \mathbb{C}$ so that $F : \mathbb{D} \to \mathbb{D}$ is a bijection.

Problem 2. Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Problem 3. Find $\displaystyle\int_0^\infty \frac{1}{1 + x^6} dx$.

Problem 4. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and there are two distinct $z_1, z_2 \in \mathbb{D}$ such that $f(z_1) = z_1$ and $f(z_2) = z_2$. Prove that $f(z) = z$ for all $z \in \mathbb{D}$.

Problem 5. How many distinct zeros does the function $f(z) = z^3 + 3z + 1$ have in the set $\{z : 1 < |z| < 2\}$?

Problem 6. Describe a holomorphic isomorphism between regions $A = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$ and $B = \{z : \mathrm{Re}(z) > 0, 0 < \mathrm{Im}(z) < 1\}$.

Problem 7.

(a) Prove that if $\mathcal{F}$ is a normal family of holomorphic functions on an open set A, then $\mathcal{G} = \{f' : f \in \mathcal{F}\}$ is also a normal family of holomorphic functions on A.

(b) Give an example where $\mathcal{G}$ is a normal family and $\mathcal{F}$ is not a normal family.