TOPOLOGY PRELIM, JANUARY 2022

Convention

- Let A and B be two sets. We denote $A \setminus B = \{x \in A; x \notin B\}$.
- Unless otherwise indicated, the space $\mathbb{R}^n$ and its subsets given below are endowed with the standard topology.

1. Let X be a topological space, and let $\{A_\alpha\}_{\alpha \in I}$ be a family of subsets in X. Is it always true that
$$\overline{\bigcup_{\alpha \in I} A_\alpha} = \bigcup_{\alpha \in I} \overline{A_\alpha} \ ?$$
Here $\overline{A}$ denotes the closure of A in X. Prove your assertion.

2. Let $B = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$ and $S = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$. Does there exist a continuous map $f : B \to S$ such that the restriction $f|_S : S \to S$ is homotopic to the identity map on S? Prove your assertion.

3. Let $\mathscr{T}$ be a collection of subsets of $\mathbb{R}^2$ which consist of the empty set and the sets of the form $\mathbb{R}^2 \setminus \{\text{at most finitely many straight lines}\}$.
   (i) Show that $\mathscr{T}$ defines a topology on $\mathbb{R}^2$.
   (ii) Is $(\mathbb{R}^2, \mathscr{T})$ a Hausdorff space? Prove your assertion.

4. Let $E = \{(x,y,z) \in \mathbb{R}^3; z^2 = x^2 + y^2 - 9\}$ with the subspace topology of $\mathbb{R}^3$. Find a universal covering space $\tilde{E}$ and an explicit covering map $p : \tilde{E} \to E$.

5. Let $\mathscr{M}$ be the set of 2×2 real matrices with the topology obtained by regarding $\mathscr{M}$ as $\mathbb{R}^4$. Let
$$\mathscr{P} = \{A \in \mathscr{M}; A^T A = I_2\}$$
endowed with the subspace topology, where A^T denotes the transpose of A, and I_2 is the 2×2 identity matrix.
   (i) Show that $\mathscr{P}$ is compact.
   (ii) Is $\mathscr{P}$ connected? Prove your assertion.

6. Let Y be the space $\mathbb{RP}^2$ with two distinct points removed. Find the fundamental group $\pi_1(Y)$.