Preliminary Examination - Numerical Analysis - January, 2022

Instructions: Solve 6 out of 8 problems. Any results that your responses rely upon must be stated clearly.

Notation:

- The set of all real numbers is denoted by $\mathbb{R}$.
- The set of all complex numbers is denoted by $\mathbb{C}$.
- $\mathbb{P}^k[a,b]$ is the vector space of all real polynomials $p : [a,b] \to \mathbb{R}$ of degree k or less.
- Bold font for column vectors, *e.g.*
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$
in which case we say $\mathbf{x} \in \mathbb{R}^n$ ($\mathbf{x} \in \mathbb{C}^n$) if $x_i \in \mathbb{R}$ ($x_i \in \mathbb{C}$) for $1 \le i \le n$.
- The transpose of $\mathbf{x}$ is denoted by $\mathbf{x}^\top$.
- Double-bar font for matrices, *e.g.* $\mathbb{A}$.

1. Given an odd integer $N \ge 1$, let $x_n = 2\pi n/N$ and consider a data set $\{(x_n, f_n)\}_{n=0}^{N-1}$. There is a trigonometric interpolant

$$\psi(x) = \frac{A_0}{2} + \sum_{n=1}^{M} \left(A_n \cos(nx) + B_n \sin(nx)\right)$$

satisfying $\psi(x_n) = f_n$ for $0 \le n \le N-1$, where $N = 2M+1$. If $f(x) = \sin(x)\cos(x+1)$ and $f_n = f(x_n)$ for $n = 0, 1, \ldots, N-1$, prove that $f(x)$ and $\psi(x)$ are identical when N is large enough.

2. Let a certain nonsingular matrix $\mathbb{A}$ of size $n \times n$ have entries in row i and column j denoted by $a_{i,j}$. The only non-zero entries of $\mathbb{A}$ are $a_{i,i} = 2$, for $1 \le i \le n$, along with $a_{i,i+1} = a_{i+1,i} = -1$, for $1 \le i \le n-1$. Given any $\mathbf{x} \in \mathbb{R}^n$, denote by $\|\cdot\|$ the Euclidean vector norm, so that $\|\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{x}$. Let the condition number of $\mathbb{A}$, say $\kappa(\mathbb{A})$, be defined relative to some matrix norm that is consistent with $\|\cdot\|$. Prove that $\kappa(\mathbb{A}) \to \infty$ as $n \to \infty$.

3. An n-point quadrature rule will be found in the form

$$I(f) = \sum_{i=1}^{n} w_i f(x_i) \approx \int_{-1}^{1} x^2\, f(x)\, dx$$

where we require $x_i \in [-1,1]$ for $1 \le i \le n$ to be distinct points. Derive the quadrature rule of this form with the smallest possible value of n such that

$$I(f) = \int_{-1}^{1} x^2 f(x)\, dx, \ \forall f \in \mathbb{P}^3[-1,1].$$

4. Write out the composite trapezoidal rule to estimate $\int_a^b f(x)\, dx$, $a < b$, using a partition $x_j = a + j \cdot h$, $j = 0, 1, \ldots, N$ and $h = (b-a)/N$. Prove that the quadrature error vanishes as $N \to \infty$, given that f is continuously differentiable everywhere on $[a,b]$.

5. Let $s_\Delta : [a,b] \to \mathbb{R}$ be a cubic spline function relative to the partition $\Delta = \{x_0, x_1, \ldots, x_N\}$ of $[a,b]$, where $x_0 = a$, $x_N = b$ and $x_{i-1} < x_i$ for $1 \le i \le N$. Assume natural spline conditions,

$s''_\Delta(a) = 0 = s''_\Delta(b)$. Also, assume that the third derivative is a uniform constant across intervals, say $s'''_\Delta(x) = C$ on (x_{i-1}, x_i) for $1 \le i \le N$ with C independent of i. Find all such $s_\Delta(x)$ on $[a, b]$. You must provide justification that your answer is correct, not just state $s_\Delta(x) =$(formula).

6. Given the matrix $\mathbb{A}$ below, calculate an upper-triangular matrix $\mathbb{R}$ for a QR-factorization $\mathbb{A} = \mathbb{Q}\mathbb{R}$ by using Householder matrices. Do not calculate $\mathbb{Q}$, but formulas for each Householder matrix must be shown, specifying numerical values for all quantities in the formulas.

$$\mathbb{A} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

7. Denote by $\tilde{\mathbb{P}}^k[-1, 1]$ the set of all monic polynomials (lead coefficient equals 1) with degree k or less. Given any non-negative integer n and $f \in \tilde{\mathbb{P}}^{n+1}[-1, 1]$, let $p \in \mathbb{P}^n[-1, 1]$ satisfy $p(x_j) = f(x_j)$ with $x_j = \cos((2j - 1)\pi/(2n + 2))$, for $1 \le j \le n + 1$. We introduce two norms:

$$\|f\|_\infty := \max_{-1 \le x \le 1} |f(x)|$$

$$\|f\|_w = \sqrt{\int_{-1}^{1} \frac{|f(x)|^2}{\sqrt{1 - x^2}}\, dx}.$$

Prove that

$$\|f - p\|_\infty < \|f - p\|_w.$$

8. Denote by $\|\cdot\|$ the Euclidean vector norm on $\mathbb{C}^n$, and let M_n be the space of all complex matrices of size $n \times n$. Given any $\mathbb{A} \in M_n$, denote the induced matrix norm by

$$\|\mathbb{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbb{A}\mathbf{x}\|.$$

Next, let $\mathbb{A} \in M_n$ be a fixed Householder matrix, and assume $\mathbb{B} \in M_n$ is given such that $\|\mathbb{B} - \mathbb{A}\| = 1/3$.
(a) Prove that $\mathbb{B}$ is nonsingular.
(b) If $\mathbb{A}\mathbf{x} = \mathbb{B}\mathbf{y}$, prove that $2\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}\|$.