Applied Math Prelim August 2022

1. (20 pts) Let X, Y be normed linear spaces and $A : X \to Y$ be a linear operator. Show that A is bounded if and only if A maps some non-empty open set in X into a bounded set in Y.

2. Let $\{\mathbf{e_1}, \cdots, \mathbf{e_n}, \cdots\}$ be an orthonormal sequence in an inner product space X. Let

$$Ax = \sum_n \lambda_n < x, \mathbf{e_n} > \mathbf{e_n}$$

   where $0 < \inf |\lambda_n| \leq \sup |\lambda_n| < \infty$.

   (a) (8 pts) Prove the series defining Ax converges.

   (b) (8 pts) Prove that A is a bounded linear operator.

   (c) (8 pts) Prove that A is not compact.

   (d) (6 pts) Find eigenvalues and eigenvectors of A.

3. (10 pts) Let f be a differentiable map between normed linear spaces. Let y_0 be a point in the target space such that f' is invertible at each point of $f^{-1}\{y_0\}$. Prove that $f^{-1}\{y_0\}$is a discrete set.

4. (15 pts) State and prove an existence and uniqueness theorem that applies to the initial value problem

$$\begin{cases} x' + \sin x = t \ , \\ x(0) = 1 \ . \end{cases}$$

5. (10 pts) Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $f \geq 0$ and $\int_{\mathbb{R}^n} f(x)dx = 1$. Let $f_j(x) = j^n f(jx)$ for $j = 1, 2 \cdots$. Show $\tilde{f}_j \to \delta$ in the distributional sense. Here $\tilde{f}_j(\phi) = \int_{\mathbb{R}^n} f_j(x)\phi(x)dx$ for all test function ϕ. Does your proof work if f is a sign changing function? Explain.

6. (15 pts) Let X be a Banach space, $A : X \to X$ be a compact linear operator. If $I + A$ is surjective, show that $I + A$ is injective.