Instructions and notation:

(i) Give full justifications for all solutions in the exam booklet. Four completely correct solutions will guarantee a Ph.D. pass.

(ii) Clearly state which theorems you are using and verify their assumptions.

(iii) For \( p \in \mathbb{C} \) and \( R > 0 \), \( B(p, R) = \{ z \in \mathbb{C} : |z - p| < R \} \) and \( \bar{B}(p, R) = \{ z \in \mathbb{C} : |z - p| \leq R \} \).

1. Evaluate the following integral and justify your answer
\[
\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} \, dx.
\]

2. Prove that the equation \( z = 2 - e^{-z} \) has exactly one root in \( H = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \).

3. Let \( f \) be an entire function such that \( f(\mathbb{C}) \cap L = \emptyset \) for some line \( L \). Show that \( f \) is a constant function.

4. Find all holomorphic functions \( f : \mathbb{C} \to \mathbb{C} \) which satisfy
\[
\int_0^{2\pi} |f(re^{it})| \, dt \leq r^\pi \text{ for all } r > 0.
\]

5. Let \( p \in \mathbb{C}, R > 0 \), and let \( f : \bar{B}(p, R) \to \mathbb{C} \) be a continuous function which is holomorphic in \( B(p, R) \). Let \( \gamma(t) = p + Re^{it}, t \in [0, 2\pi] \). Show that
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \text{ for all } z \in B(p, R).
\]

6. Prove that if \( f \) has a non-removable isolated singularity at \( p \in \mathbb{C} \) then \( e^f \) has an essential singularity at \( p \).

7. Let \( U \subseteq \mathbb{C} \) be a connected and simply connected set with \( p, q \in U, p \neq q \). Let \( f, g : U \to U \) be bijective holomorphic functions such that
\[
f(p) = g(p) \text{ and } f(q) = g(q).
\]
Show that \( f = g \).