Complex Analysis Preliminary Exam, August 2022

Instructions and notation:

(i) Give full justifications for all solutions in the exam booklet. Four completely correct solutions will guarantee a Ph.D. pass.

(ii) Clearly state which theorems you are using and verify their assumptions.

(iii) For $p \in \mathbb{C}$ and $R > 0$, $B(p, R) = \{z \in \mathbb{C} : |z - p| < R\}$ and $\bar{B}(p, R) = \{z \in \mathbb{C} : |z - p| \leq R\}$.

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1. Evaluate the following integral and justify your answer

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} \, \mathrm{d}x.$$

2. Prove that the equation $z = 2 - e^{-z}$ has exactly one root in $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

3. Let f be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$ for some line L. Show that f is a constant function.

4. Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ which satisfy

$$\int_0^{2\pi} |f(re^{it})| \, \mathrm{d}t \leq r^{\pi} \text{ for all } r > 0.$$

5. Let $p \in \mathbb{C}, R > 0$, and let $f : \bar{B}(p, R) \to \mathbb{C}$ be a continuous function which is holomorphic in $B(p, R)$. Let $\gamma(t) = p + Re^{it}, t \in [0, 2\pi]$. Show that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, \mathrm{d}w \text{ for all } z \in B(p, R).$$

6. Prove that if f has a non-removable isolated singularity at $p \in \mathbb{C}$ then e^f has an essential singularity at p.

7. Let $U \subsetneq \mathbb{C}$ be a connected and simply connected set with $p, q \in U, p \neq q$. Let $f, g : U \to U$ be bijective holomorphic functions such that

$$f(p) = g(p) \text{ and } f(q) = g(q).$$

Show that $f = g$.