Loss Models Prelims for Actuarial Students
Friday, August 26, 2022, 9am–1pm
MONT 214

Instructions:

1. There are five (5) questions here and you are to answer all five. Each question is worth 20 points.
2. Hand-held calculators are permitted.
3. Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
4. Please write legibly. Points will be deducted for incoherent, incorrect, and/or irrelevant statements.

Question No. 1:

Claims arrival process can be described according to type of claim i with N_i, the number of claims for the i-th type, and claim size x_i, the fixed amount of claim for the i-th type, $i = 1, \ldots, m$. The probability that the claim is of type i is denoted by p_i, $i = 1, \ldots, m$. In this case, the total aggregate claims can be represented as

$$S = x_1 N_1 + \cdots + x_m N_m.$$

You are given that total number of claims N has a Poisson distribution with mean λ, where

$$N = N_1 + \cdots + N_m.$$

(a) Explain why

$$\Pr(N_1 = n_1, \ldots, N_m = n_m | N = n) = \frac{n!}{n_1! \cdots n_m!} p_1^{n_1} \cdots p_m^{n_m},$$

a multinomial distribution. In particular, prove that

$$\Pr(N_i = n_i | N = n) = \binom{n}{n_i} p_i^{n_i} (1 - p_i)^{n - n_i},$$

for $i = 1, \ldots, m$, a binomial distribution.

(b) Prove that the random variables $N_1, \ldots, N_m$ are independent and each has Poisson distribution with respective parameters $\lambda_i = p_i \lambda$, for $i = 1, \ldots, m$. Hint: Prove this for the case when $m = 2$ and explain why this easily generalizes to the case where $m > 2$.

(c) Now, consider the following application: assume that the number of claims during a week is a Poisson with mean 400. Claim size arrives at either \$300 with probability 0.60, or \$150 with probability 0.40. Calculate the mean and variance of the total aggregate claims for a week.

Question No. 2:

Consider a collective risk model $S = \sum_{i=1}^{N} X_i$, where $X_1, X_2, \ldots$ are i.i.d. random variables with the same distribution as X and is independent of the claim frequency N.

The claim frequency N follows a Poisson distribution with mean 5, and the claim severity X is given by

$$X = (Y - c)^+,$$

where $c > 0$ and Y follows an exponential distribution with mean 10, i.e.,

$$f_Y(y) = \frac{1}{10} e^{-y/10}, \quad y > 0.$$

(a) It is given that $\mathrm{E}[S] = 45$, determine c.

(b) Calculate the variance of S.

(c) Now suppose that an insurer has sold the same policy to m independent customers, in which the risk of each policy S_j, $j = 1, \ldots, m$, follows the same distribution as S and the premium of each policy is based on the expected value principle with loading 10%. What is the minimum m such that the insurer makes a positive profit with at least 95% probability?

Hint: You may apply normal approximation. For a standard normal $Z \sim \mathcal{N}(0, 1)$, $\Pr(Z \leq 1.64) = 0.9495$ and $\Pr(Z \leq 1.65) = 0.9505$.

Question No. 3:

A Cox proportional hazards model was used to study the claims on two groups of insurance policies. A single (binary) covariate z was used with $z = 0$ for a policy in Group A and $z = 1$ for a policy in Group B.

You are given the following observed claims from each of these two groups:

Group A:	275, 325, 400, 520
Group B:	200, 225, 250, 300

The baseline survival function is given by:

$$S(x) = \Pr(X > x) = \left(\frac{200}{x}\right)^{\alpha}, \quad x > 200, \alpha > 0.$$

(a) Calculate the maximum likelihood estimate of the coefficient β of the proportional hazards model.

(b) Without performing the calculations, explain, in detail, the procedure or steps you would take to calculate an estimated standard error for this maximum likelihood estimate.

Question No. 4:

Suppose that conditionally on λ, claims on an insurance policy denoted by $X_1, \ldots, X_n$ are distributed as Poisson with mean λ.

Let the prior distribution of λ be a Gamma, with parameters a and b, with expression of density given as in the appendix.

(a) Prove that the posterior distribution of λ, given the observed data $x_1, \ldots, x_n$, is also Gamma distribution. Find expressions for its parameters.

(b) Show that the resulting Bayes estimator can be expressed as the weighted combination of the sample mean and the prior mean.

continued: Now, suppose that the prior distribution of λ has the Lindley distribution with probability density function:

$$\pi(\lambda) = \frac{a^2}{a+1}(\lambda + 1)e^{-\lambda a},$$

where parameter $a > 0$.

(c) Show that the density of the Lindley distribution given above can be expressed as a mixture of two Gamma distributions expressed as:

$$\pi(\lambda) = w_1 \pi_1(\lambda) + w_2 \pi_2(\lambda).$$

Specify w_1 and w_2, which are weights that depend only on a and show that $w_1 + w_2 = 1$. Show that $\pi_1(\lambda)$ and $\pi_2(\lambda)$ are each density functions of Gamma distributions. Specify their respective parameters.

(d) Use all the results above to find the expression for the Bayes estimator in terms of a and the observed data $x_1, \ldots, x_n$.

Question No. 5:

Suppose the probability density function (pdf) of losses X is given by

$$f_X(x) = \frac{\alpha}{x^{\alpha+1}}, \quad x > 1, \alpha > 0.$$

A random sample of 5 losses is observed with values 2, 2, 4, and two losses exceeding 5.

(a) Calculate the maximum likelihood estimate of α.
Please round the final result to one decimal place, and use it for the rest of the questions.

(b) Calculate $\mathrm{VaR}_\delta(X)$, where $\delta \in (0, 1)$.

—— end of exam ——

APPENDIX

A random variable X is said to have a Gamma distribution with rate parameter $a > 0$ and shape parameter $b > 0$ if its probability density function has the form

$$f(x) = \frac{1}{\Gamma(b)} a^b x^{b-1} e^{-ax}, \quad x > 0.$$

Its mean and variance are, respectively,

$$\mathrm{E}[X] = \frac{b}{a} \quad \text{and} \quad \mathrm{Var}[X] = \frac{b}{a^2}.$$

Note that when $b = 1$, this results in an Exponential distribution.