Real Analysis Preliminary Exam, August 2022

Instructions and notation:
(i) Give full justifications for all answers in the exam booklet.
(ii) The set difference of two sets $A, B$ is defined as $A \Delta B := (A \setminus B) \cup (B \setminus A)$. The characteristic function of a set $A$ is denoted by $1_A$. Lebesgue measure on $\mathbb{R}^n$ is denoted by $L^n$ and $dx$ corresponds to Lebesgue integration in $\mathbb{R}$.

1. (10 points) Let $(X, \mathcal{B}, \mu)$ be a measure space. For any $A, B \in \mathcal{B}$, let
   
   \[ d(A, B) = \mu(A \Delta B) = \int |1_A - 1_B| \, d\mu. \]

   (a) (5 points) Show that
   
   \[ d(A, C) \leq d(A, B) + d(B, C) \]
   
   for all $A, B, C \in \mathcal{B}$.

   (b) (5 points) Let $A_1, B_1, A_2, B_2, \ldots \in \mathcal{B}$. Show that
   
   \[ d \left( \bigcup_{n \geq 1} A_n, \bigcup_{n \geq 1} B_n \right) \leq \sum_{n=1}^{\infty} d(A_n, B_n). \]

2. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space and $g : X \to [0, \infty]$ a measurable function. Define
   
   \[ \nu(E) = \int_E g \, d\mu \quad \text{for all } E \in \mathcal{F}. \]

   Show that $\nu$ is a measure, and then show that
   
   \[ \int f \, d\nu = \int fg \, d\mu \]
   
   for any measurable function $f : X \to [0, \infty]$.

3. Compute the limit
   
   \[ \lim_{n \to \infty} \int_0^n \left( 1 + \frac{x}{n} \right)^n \cos x \, dx. \]

4. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Let $p, q \in [1, \infty]$ with $p \leq q$. Show that
   
   \[ \|f\|_p \leq \|f\|_q, \]
   
   for all $p, q \in [1, \infty]$ with $p \leq q$ and all functions $f \in L^q(\mu)$.

5. (10 points) Let $(X, \mathcal{F}, \mu)$ be a measure space, $(Y, \mathcal{G})$ a measurable space, and $\phi : X \to Y$ a measurable map. Define
   
   \[ \nu(A) = \mu(\phi^{-1}(A)) \quad \text{for all } A \in \mathcal{G}. \]

   Show that $\nu$ is a measure and
   
   \[ \int f \, d\nu = \int f \circ \phi \, d\mu \]
   
   for any measurable function $f : Y \to [-\infty, \infty]$ for which the integral on the right hand side is defined.
6. (10 points) Let $E$ and $F$ be Borel subsets of $\mathbb{R}^2$, such that

$$\mathcal{L}^1(E_x) = \mathcal{L}^1(F_x) \quad \text{for all } x \in \mathbb{R},$$

where $A_x = \{ y \in \mathbb{R} : (x, y) \in A \}$ denotes the $x$-section of any $A \subset \mathbb{R}^2$. Show that $\mathcal{L}^2(E) = \mathcal{L}^2(F)$.

7. (10 points) Let $\mu_1, \mu_2, \ldots$ be a sequence of Radon measures on a locally compact Hausdorff space $X$. Suppose also that

$$\lim_{n \to \infty} \int f \, d\mu_n$$

exists in $\mathbb{R}$ for every $f : X \to \mathbb{R}$ that is continuous of compact support. Show that there is a Radon measure $\mu$ on $X$ such that

$$\int f \, d\mu = \lim_{n \to \infty} \int f \, d\mu_n \quad \text{for all } f \in C_c(X).$$