MATH 313 August 2005 PRELIMINARY EXAMINATION

INSTRUCTIONS: Answer three out of four questions. You do not have to prove results which you rely upon, just state them clearly.

Good luck!

Q1) (a) Suppose that $A^{(1)} = A$ is an invertible $n \times n$ matrix and that the Gaussian elimination algorithm with partial pivoting applied to $A^{(1)}$ produces the upper triangular matrix $A^{(n)}$. As usual, let $A^{(k)}$ be the renamed array following any necessary row intrechanges before the k–th major step of the elimination so that

$$a_{i,j}^{(k+1)} = \begin{cases} a_{i,j}^{(k)}, & \textit{when } \text{i} \leq k,\ 1 \leq j \leq n, \\ 0 & \textit{when } \text{i} \geq k+1,\ 1 \leq j \leq k, \\ a_{i,j}^{(k)} - a_{i,k}^{(k)} a_{k,j}^{(k)} / a_{k,k}^{(k)}, & \textit{when } \text{i,j} \geq k+1. \end{cases}$$

Show that the total number of multiplication and division operations needed to reduce $A^{(1)}$ to $A^{(n)}$ is $(n^3 - n)/3$. [Hint: Recall that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.]

(b) Suppose that all the leading principal minors of A are positive. Show that A has an LU–factorization with unit diagonal entries in L and positive diagonal entries in U.

(c) Suppose now that no partial pivoting is necessary and that $A^{(1)} = \left(a_{i,j}^{(1)}\right)$ is tridiagonal, that is, $a_{i,j}^{(1)} = 0$ when $|i - j| > 1,\ 1 \leq i, j \leq n$. Show that each of $A^{(1)}, \ldots, A^{(n)}$ is tridiagonal.

(d) Suppose that A is an $n \times n$ invertible matrix which admits an LU–factorization without pivoting. Partition A into:

$$A = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right],$$

with $A_{1,1}$ being a $(k-1) \times (k-1)$ matrix. Knowing that $A_{1,1}$ is invertible (why?), show that the current active array which is the $(n-k+1) \times (n-k-1)$ matrix $A_k = \left(a_{i,j}^{(k)}\right)$, $i, j = k, \ldots, n$ is given by:

$$A_k = A_{2,2} - A_{2,1} A_{2,2}^{-1} A_{1,2}.$$

Assume now that in addition to A being invertible, A is Hermitian. Use this formula to deduce that A_k is also Hermitian, $k = 1, \ldots, n$.

Q2)(a) Let $a \leq x_0 < x_1 < .... < x_n \leq b$ be $n+1$ distinct numbers. Show that there exists a unique polynomial of degree at most n^{th} which passes through (x_i, y_i), $i = 0, 1, ..., n$.
(b) Show that there exist unique numbers γ_i, $i = 0, 1, ..., n$, such that

$$\sum_{i=0}^{n} \gamma_i P(x_i) = \int_a^b P(x)\, dx$$

for all polynomials P with degree(P) $\leq n$.
(c) Divide the interval $[0, 1]$ into m equal length subintervals and implement the composite integration rule of (b) with $n = 2$ and some fixed set of support points. For large m, what is the relationship between m and the approximation error,

$$\left| \int_0^1 f(x)dx - \sum_{j=1}^{m} \sum_{i=0}^{2} \gamma_{i,j} f(x_{i,j}) \right|$$

where f is a C^∞ function? (Hint: state and use the Peano Theorem).

Q3) Let $w(x)$ be a positive continuous function on $[a,b]$. For $j = 1, 2, \ldots$, let $p_j(x)$ be the corresponding monic orthogonal polynomial of degree j, i.e.,

$$p_j(x) = x^j + a_1 x^{j-1} + \cdots + a_j,$$

such that $(p_j, p_k) = \int_a^b w(x) p_j(x) p_k(x) dx = 0$ if $j \neq k$. In particular $p_0(x) = 1$.
(a) Prove that the roots $x_1, .., x_n$ of $p_n(x)$ are real, simple and lie in (a,b).
(b) Prove that $p_n(x)$ satisfy a three term recurrence relation, i.e.,

$$p_{i+1}(x) = (x - \delta_{i+1}) p_i(x) - \gamma_{i+1}^2 p_{i-1}(x), \quad i \geq 0,$$

where $p_{i-1} = 0$, $\gamma_1 = 0$, and

$$\delta_{i+1} = \frac{(xp_i, p_i)}{(p_i, p_i)}, \quad i \geq 0, \quad \gamma_{i+1}^2 = \frac{(p_i, p_i)}{(p_{i-1}, p_{i-1})}, \quad i \geq 1.$$

(c) For $a = -1; b = 1; w(x) = 1$; find $p_1(x)$ and $p_2(x)$.

Q4) Consider the $k \times k$ matrix

$$A_k = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 0 & \sqrt{2} \\ & & & & & \sqrt{2} & 0 \end{bmatrix}$$

- Set

$$p_0 = 2, \qquad p_1(x) = 2x, \tag{1}$$

and define the family of polynomials $\{p_k(x)\}$ by

$$p_k(x) = \det(2xI - A_k). \tag{2}$$

For $k = 3, 4, \ldots$ derive the recurrence relations of the form

$$p_k(x) = (\alpha_k x - \beta_k)p_{k-1}(x) - \gamma_k p_{k-2}(x), \tag{3}$$

i.e., find $\{\alpha_k, \beta_k, \gamma_k\}$ in (3) explicitly.

(Hint: Expand the determinant in (2) along the first row.)

- The family of polynomials $\{p_k(x)\}$ is closely related to one of the classical families of orthogonal polynomials. Identify the latter orthogonal polynomials $\{Q_k(x)\}$ by the name, and explain the relationship between $\{p_k(x)\}$ and $\{Q_k(x)\}$.

  (Hint: The two families $\{p_k(x)\}$ and $\{Q_k(x)\}$ slightly differ from each other. To state the relationship between them look not only at the recurrence relations (3) for $\{p_k(x)\}$, but also at first two polynomials $p_0(x), p_1(x)$ in (1).)

- Use the obtained relationship between $\{p_k(x)\}$ and $\{Q_k(x)\}$ to write down an **explicit** formula of the form

$$p_k(x) = \text{"an explicit expression in terms of } k \text{ (and } \textbf{not} \text{ involving } p_{k-1}(x), p_{k-2}(x), p_{k-3}(x), \ldots\text{"} \tag{4}$$

  (Hint: the right hand side of (4) should contain trigonometric functions.)

- You wrote down the formula (4) using the relationship between $\{p_k(x)\}$ and $\{Q_k(x)\}$. Now provide an independent proof of it, deducing (4) from (1) and (3).

- Use the formula (4) to derive explicit expressions for the eigenvalues of the matrix A_k.