307

Topology Preliminary Exam 06

August, 2006

1. A topological space X is called a "Lindeöf space" if every open cover has a countable subcover.

   a) Show that every second countable space is a Lindeöf space.

   b) Given an example of a Lindeöf space that is not second countable.

2. Consider the capital letters of the alphabet, as below, in the sans serif style with no adornments.

   ABCDEFGHIJKLMNOPQRSTUVWXYZ

   Each letter is a topological space, with the subspace topology inherited from R^2.

   a) Prove that X is not homeomorphic to Y.

   b) Give an explicit homeomorphism from O to D

   c) Consider the equivalence relation "is homeomorphic to" on the set of these letters. What are its equivalence classes?

3. A metric space is called "totally bounded" if for every $\epsilon > 0$, there is a finite covering of the space by ϵ balls. Prove or disprove that a totally bounded metric space is separable.

4. Let $\mathbf{T}$ be the collection of sets $U's \subset R^2$ such that $U's$ are either the empty set or satisfy that for each $(x, y) \in U$, there is an open line segment in each direction about (x, y) that is contained in U.

   a) Show $\mathbf{T}$ is a topology on R^2.

   b) Compare $\mathbf{T}$ with the standard topology; that is, is $\mathbf{T}$ weaker, stronger, the same or none of these.

   c) Let L denote a straight line in R^2. Compare the subspace topologies on L induced by these two topologies.

   d) Let S denote a circle in R^2. Compare the subspace topologies on S induced by these two topologies.

5. Let X_α be a family of topological spaces, where α runs over an index set. Let X be a set and let $f_\alpha : X \to X_\alpha$ be a family of functions.

   a) Define (describe) the smallest topology on X such that each $f_\alpha : X \to X_\alpha$ becomes continuous.

   b) Suppose that X has the topology given in (a). Given a topological space Y, show that a function $f : Y \to X$ is continuous if and only if $f_\alpha \circ f : Y \to X_\alpha$ is continuous for each α.

6. A topological space X is said to be **locally compact at a point** $x \in X$ if there exists an open set U and a compact set C such that $x \in U \subseteq C \subseteq X$. The space X is **locally compact** if it is locally compact at every point. Prove or disprove the following statements.

a) Every compact space is locally compact.

b) R^n is locally compact.

c) R^∞ with the product topology is locally compact.

d) R^∞ with the box topology is locally compact.

Good Luck!!