303

Real Analysis Qualifying Exam
Summer 2006

1. Prove

$$\sum_{k=1}^{\infty} \frac{1}{(p+k)^2} = \int_0^1 \frac{x^p \log x^{-1}}{1-x} dx$$

for $p > 1$. Justify each step, in particular indicate why certain improper Riemann integrals are Lebesgue integrals.

2. Give an example of a sequence of functions that converges in L_1 but not almost everywhere (a,e.). Show that, on the other hand, if f_n, $n \in \mathbf{N}$, and f are in L_1 and $f_n \to f$ in L_1 fast enough so that $\sum_n \int |f_n - f| < \infty$, then $f_n \to f$ a.e.

3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $f \in L_1(X, \mu)$, $f \geq 0$. Show that the subgraph of f,

$$G_f := \{(x, y) \in X \times [0, \infty] : y \leq f(x)\}$$

is $\mathcal{M} \times \mathcal{B}_{\mathbf{R}}$-measurable and

$$(\mu \times m)(G_f) = \int f d\mu,$$

where m is Lebesgue measure (the integral of a non-negative function is the area under its graph, above the 'x-axis').

4. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin x^{-2} & \text{if } 0 < x \leq 1 \end{cases}.$$

Determine whether this function is of bounded variation on $[0, 1]$ and whether it is absolutely continuous on $[0, 1]$. Determine the same on $[\delta, 0]$ for any $0 < \delta < 1$. Justify your answers.

5. Show that if $f \in L_p(\mathbf{R}) \cap L_\infty(\mathbf{R})$ for some $p \geq 1$ then $f \in L_q(\mathbf{R})$ for all $q > p$ and

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q.$$

Hint: It suffices to consider $\|f\|_\infty = 1$ and $|f(x)| \leq 1$ for all x, and, in this case, it will help to look at the functions $f_\delta := (|f| \wedge (1 - \delta))/(1 - \delta)$ for suitable $0 < \delta < 1$ for the inequality in one direction.