Abstract Algebra Prelim Aug. 2008

1. a) Show every nontrivial finite p-group has a nontrivial center.

   b) Use part a to show for every nontrivial finite p-group G that the commutator subgroup $[G, G]$ is a proper subgroup.

2. A *character* of a finite abelian group G is a homomorphism $\chi\colon G \to \mathbf{C}^\times$. The characters of G form a group under pointwise multiplication: $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for $g \in G$. The group of characters of G is denoted $\widehat{G}$.

   a) If G is a finite cyclic group of order n, prove the character group $\widehat{G}$ is also cyclic of order n.

   b) If G is a finite cyclic group, prove G and $\widehat{G}$ are isomorphic using an explicit isomorphism.

3. Let V be a nonzero vector space over a field K. Use Zorn's lemma to prove V has a basis. (Your proof must be applicable to all nonzero vector spaces over K.)

4. Let R be a domain.

   a) If R is a field, prove $R[X]$ is a PID.

   b) If $R[X]$ is a PID, prove R is a field. (Hint: As a special case, can you think of a nonprincipal ideal in $\mathbf{Z}[X]$?)

5. Let $R = A \times B$ be a product of two nonzero commutative rings.

   a) If $\mathfrak{p}_A$ is a prime ideal in A and $\mathfrak{p}_B$ is a prime ideal in B, show $\mathfrak{p}_A \times B$ and $A \times \mathfrak{p}_B$ are prime ideals in R.

   b) Prove all prime ideals in R have one of the two forms indicated in part a. (Hint: Start by showing any prime ideal in R contains either $(1, 0)$ or $(0, 1)$.)

6. Give examples as requested, with brief justification.

   a) A solvable group that is not nilpotent.

   b) A linear operator on some complex vector space which is not diagonalizable.

   c) An algebraic property distinguishing the rings $\mathbf{Z}[i]$ and $\mathbf{Z}[\sqrt{-5}]$ (so these rings are not isomorphic).

   d) A construction in algebra that can be characterized by a universal mapping property. (State the universal mapping property, but you don't have to show it characterizes the construction.)