Math 310 Preliminary Exam
Fall 2008

Name ____

Signature____

1. (20 pts)

   (a) Let $\tau_0 \in (0,1)$. Find the Green's Function for

   $$\begin{aligned} y'' + y &= \delta(t - \tau_0) \\ y'(0) &= y(1) = 0. \end{aligned}$$

   (b) Show that there exists a unique solution for

   $$\begin{aligned} y'' + y &= \lambda \tan^{-1} y + \cos x \\ y'(0) &= y(1) = 0 \end{aligned}$$

   for $|\lambda|$ sufficiently small.

2. (20 pts) Let T be a compact operator on a Hilbert space $\mathcal{H}$ and $\{\varphi_n : n \in N\}$ be an orthonormal system of $\mathcal{H}$.

   (a) (5 pts) Show $\varphi_n \rightharpoonup 0$ weakly. Explain why this gives an example of weakly convergent sequence which is not strongly convergent.

   (b) (5 pts) Using a. or otherwise, show $\|T\varphi_n\| \longrightarrow 0$

   (c) (10 pts) Let λ_n be a sequence of complex numbers. Then operator S defined by $Sf = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi_n$ is compact iff $\lim_{n \longrightarrow \infty} \lambda_n = 0$.

3. (20 pts)

   (a) Let f be an operator on a Banach space X, give the definition of f being Fréchet differentiable at a point $x \in X$.

   (b) Define $f : C[0,1] \longrightarrow C[0,1]$ by $[f(x)](t) = x(t) + \int_0^1 (x(st))^2 \, ds$. Compute $f'(x)$.

4. (20 pts) Let K be a compact operator on a Banach space. If $I + K$ is injective, then it is surjective.

5. (20 pts) Let $T(\varphi) = \varphi(-1) + \varphi(1)$ for every $\varphi \in \mathcal{D}(R)$.

   (a) Show T is a distribution.

   (b) Find ∂T the distributional derivative of T.