COMPLEX ANALYSIS PRELIMINARY EXAMINATION

August 2008

Special Instructions:

(1) Justify your answers, and show your work in the examination booklet.

(2) Notation:

- $\mathbb{C}$ **the space of complex numbers**;
- $B(a,R) = \{z \in \mathbb{C} : |z-a| < R\}$;
- $D = B(0,1)$, **the unit disk in** $\mathbb{C}$;
- $\mathcal{H}(G)$, **the space of holomorphic (analytic) functions in the region** G.

1. Suppose f is entire, $f(0) = 3+4i$, and $|f(z)| \leqslant 5$ in D. Find $f'(0)$.

2. Find a conformal map (an explicit formula) of the first quadrant $\{z \in \mathbb{C} : \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$ onto D. Is it unique? Can you give some conditions to ensure that this conformal map is unique?

3. How many zeroes counting multiplicities does $z^5 + 3z^3 + 7$ have in $B(0,2)$?

4. Evaluate and **justify** your answer.
(a)

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1} dx.$$

(b)

$$\int_{0}^{2\pi} e^{e^{i\theta}} d\theta.$$

5. Suppose that f is holomorphic in D such that

$$|f(z)| \leqslant \frac{1}{1-|z|}, \text{ for any } z \in D.$$

Prove that

$$|f'(z)| \leqslant \frac{4}{(1-|z|)^2}, \text{ for any } z \in D.$$

6. Consider a family of functions in $\mathcal{H}(D)$ which are uniformly bounded at 0 and whose derivatives are uniformly bounded on compact subsets of D. Show that this family is normal. If the condition of uniform boundedness at 0 is removed, what happens then? Explain.

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