Abstract Algebra Prelim Aug. 2009

1. For $a \in \mathbf{Z}/n\mathbf{Z}$, let $\pi_a \colon \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ by $\pi_a(x) = x + a$. This is a permutation on $\mathbf{Z}/n\mathbf{Z}$.
   (a) Compute the order of $\pi_a$ as a permutation. Your answer will depend on $n$ and $a$.
   (b) Determine when $\pi_a$ is an even permutation. Your answer will depend on $n$ and $a$.

2. State and prove the Eisenstein criterion for polynomials in $\mathbf{Z}[X]$.

3. Let $R$ be a commutative ring and define $J$ to be the intersection of all maximal ideals of $R$.
   (a) Prove that $J$ is an ideal.
   (b) Let $a \in J$. Prove that for all $b \in R$, the element $1 - ab$ is invertible.

4. Let $F_1, \ldots, F_n$ be fields, where $n \geq 2$, and set $A = F_1 \times \cdots \times F_n$ (the product ring). For any subset $S$ of $\{1, \ldots, n\}$, let

$$I_S = \{(x_1, \ldots, x_n) \in A : x_i = 0 \text{ for } i \in S\},$$

   so in particular $I_\emptyset = A$ and $I_{\{1,\ldots,n\}} = \{\mathbf{0}\}$. (Smaller $S$ make larger $I_S$.)
   (a) Prove $I_S$ is an ideal in $A$ and describe the ring $A/I_S$ in terms of a product of fields.
   (b) For any $\mathbf{x} = (x_1, \ldots, x_n) \in A$, describe the principal ideal $A\mathbf{x}$ in terms of the coordinates of $\mathbf{x}$.
   (c) Show every ideal in $A$ has the form $I_S$ for some subset $S$ of $\{1, \ldots, n\}$.

5. Let $G$ be a finite group which acts on a set $X$.
   (a) Let $N = \{g \in G : gx = x \text{ for all } x \in X\}$. Show $N$ is the largest normal subgroup of $G$ contained in the stabilizer subgroup of each point in $X$.
   (b) If the action has one orbit, show for any two points $x$ and $y$ in $X$ that their stabilizer subgroups are conjugate.
   (c) Let $G = S_3$ be the permutation group on 3 elements. Give two examples of actions of $G$ on itself where
      (i) there is only one orbit,
      (ii) there is more than one orbit and the conclusion of part (b) is false.

      Provide a brief explanation of why your answers to (i) and (ii) fit the conditions.

6. Give examples as requested, with brief justification.
   (a) A nonabelian group of order 27.
   (b) A prime element of $\mathbf{Z}[i]$.
   (c) A cyclic group with exactly 8 generators.
   (d) A free module (over some ring), and a nonzero submodule which is not free.