Name: ______________________________

Math 5410 Prelim August, 7, 2009

Choose 5 out of the 6 questions.

(1a) State and prove an existence and uniqueness theorem for the equation $\frac{d^2x}{dt^2} + f(x) = 0$ with initial conditions $x(0) = a$ and $x'(0) = b$ under the assumption that f and its partial derivatives are continuous. (You can assume the Contraction Mapping Theorem).

(1b) Let $a = b = f(0) = 0$ in part (a). Can $x(t) = t^3$ be a solution to part (a)? Explain.

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au = -u'' + u$$

with $u'(0) = u'(1) = 0$.
(2b) Define $T : L^2(0, 1) \to L^2(0, 1)$ such that for any $f \in L^2(0, 1)$,

$$(Tf)(x) = \int_0^1 G(x, y) f(y)\, dy\ .$$

Explain what spectral theorem is and why it is applicable.
(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3) Let

$$U(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le 1,\ y \ge 0, \\ \\ 0, & \text{otherwise,} \end{cases}$$

Compute its distributive derivative $D_{xy}U$ in $\mathbf{R^2}$.

(4) Let H be a Hilbert space and $K : H \to H$ is a linear, bounded, compact operator. Define $A = I + K$. Show that if A is injective, then it is surjective.

(5) Let H be a Hilbert space and $A : H \to H$ is compact. Show that
(a) $x_n \rightharpoonup x$ weakly implies $Ax_n \to Ax$.
(b) The operator norm of A is attained.

(6a) Let K be a closed convex set in a Hilbert space X. Let $x \in X$ and let y be the point of K closest to x. Prove that $\mathcal{R}e\langle x-y, v-y\rangle \le 0$ for all $v \in K$, where $\mathcal{R}e$ denotes the real part.
(6b) For each x in X, we use Px to denote the point of K closest to x. Using part (a) or otherwise, prove that

$$\|Px - Pz\| \le \|x - z\| \ .$$