Topology August 2009

1. (a) Suppose A and B are connected subsets of the space X and $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected.

   (b) Let Y denote the set of all points in the plane $\mathcal{R}^2$ with at least one irrational coordinate. Prove or disprove: Y is connected.

   (c) Let $\{A_i\}$ be a sequence of connected subspaces of $\mathcal{R}^2$ such that $A_{i+1} \subset A_i$ for $i = 1, 2, 3, \ldots$. Prove or disprove: $\cap\{A_i | i \geq 1\}$ is connected.

2. Let X be a compact Hausdorff space.

   (a) Prove that X is normal.

   (b) Let $C_1 \supset C_2 \supset \ldots \supset C_n \supset C_{n+1} \supset \ldots$ be a nested sequence of closed subsets of X. Prove that $Y = \cap_{n=1}^{\infty} C_n$ is nonempty.

   (c) In part (b), make the additional assumption that each C_n is connected and then prove that Y is also connected.

3. Let $\sim$ be an equivalence relation on the compact Hausdorff space X. Let $p : X \longrightarrow X/\sim$ denote the quotient map to the set of equivalence classes $X/\sim$ equipped with the quotient topology. Recall that a subset B of X is *saturated* is $B = p^{-1}(p(B))$.

   Assume that p is a closed map.

   (a) Suppose that U is an open set in X containing a saturated set A of X. Prove that there exists a saturated open set V of X such that $A \subset V \subset U$.

   (b) If X is compact and Hausdorff, prove that $X/\sim$ is Hausdorff.

4. Recall that a space X is said to be *first countable* if at each $x \in X$ there is some countable local base. This means, given x, there is a countable collection $\mathcal{O}_x$ of open sets containing x such that whenever U is an open set containing x there exists some $V \in \mathcal{O}_x$ such that $V \subset U$.

   Let x be a point and A a subset of a first-countable space X. Prove that $x \in \overline{A}$ if and only if there exists some sequence of points in A converging to x in X.