MATH 5410, Preliminary Exam

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NAME: ____________________________ SIGNATURE: ____________________________

1. a) What is the definition of a compact linear operator from a Banach space X to itself;

b) Give an example of an operator for $X = L^2([0,1])$ which is a compact linear operator and explain why;

c) Give an example of an operator for $X = L^2([0,1])$ which is NOT a compact linear operator and explain why;

2. a) What is the definition of weak convergence of a sequence $\{x_n\}$ in a Hilbert space H;

b) Prove that a strongly convergent sequence is also a weakly convergent sequence in H;

c) Give an example of a weakly convergent sequence which is NOT strongly convergent in $L^2([0,1])$ and explain;

3. a) Give an example of a distribution which can NOT be identified with a continuous function in R and explain why.

b) Define $\delta(0)$ as a distribution;

c) If $T(\phi) = \phi(0) + \phi'(1)$ for every $\phi \in \mathcal{D}(\mathcal{R})$, find ∂T the derivative of T.

4. a) Suppose f is an operator from Banach space X to itself. Give the definition of f being Fréchet differentiable at a point $x \in X$.

b) Let $X = C[0,1]$ with sup-norm. Let $t_i \in [0,1]$ and $v_i \in C[0,1]$, and define $f(x) = \Sigma_{i=1}^{n}(x(t_i))v_i$. Prove that f is Fréchet differentiable at all points of X and find a formula for f'.

5. Find a function in $C^1[0,1]$ that minimizes the integral $\int_0^1 [(u'(t))^2 + u(t)]dt$ with constraints $u'(0) = 0$ and $u(1) = 1$.

6. Find an orthonormal basis for $L^2[0,1]$ by considering the Sturm-Liouville operator
$Ax = x'' + x$ with $x(0) = x(1) = 0$. Explain the reasons (theory) behind your method.