Real Analysis PhD Exam, August 2011

1. In this problem, prove assertions a) and b) by invoking limit theorems for integrals. Check carefully the conditions that allow you to apply these theorems.

a) $\lim_{n\to\infty} \int_0^1 \frac{n + n^k x^k}{(1+\sqrt{x})^n} dx = 0$ for all $k \in \mathbb{N}$.

b) If $f : [0, \infty) \mapsto \mathbb{R}$ is a non-negative, monotone non-increasing function which is Lebesgue integrable on $[0, \infty)$, then $\int_0^\infty f(x)dx = \lim_{n\to\infty} \int_0^{x_n} (f(x) - f(x_n))dx$ whenever $x_n \nearrow \infty$ and, as a consequence, $\lim_{x\to\infty} xf(x) = 0$.

c) Is $\lim_{x\to\infty} xf(x) = 0$ true if in b) we replace the monotonicity condition by uniform continuity? Justify your answer.

2. In this question, f_n, f are measurable functions defined on a general measure space (X, Σ, μ). Answer whether each of the following statements is true or false, provide a proof if your answer is 'true', and provide a counterexample if your answer is 'false'.

a) If $\sum_{n=1}^\infty \int |f_n - f|^p d\mu < \infty$ for some $p > 0$ then $f_n \to f$ μ-a.e.

b) $f_n \to f$ in $L_p(\mu)$ (for some $p \geq 1$) implies $f_n \to f$ μ-a.e.

3. Let $f : [0, 1] \mapsto \mathbb{R}$ satisfy the property that there exists $M < \infty$ such that $\|f\|_p \leq M$ for all $1 \leq p < \infty$, where $\| \cdot \|_p$ denotes the L^p norm for Lebesgue measure on $[0, 1]$.

a) Does it follow that $f \in L^\infty([0, 1])$?

b) What if $f \in L^p([0, 1])$ but no such cconstant M exists? (That is, are there any functions that are in $L^p([0, 1])$ for all $1 \leq p < \infty$ but not in $L^\infty([0, 1])$?)

4. Let F and G be two functions of bounded variation on $[a, b]$, $-\infty < a < b < +\infty$. Assume F is continuous and G is right continuous, and let μ_F and μ_G be the corresponding Lebesgue-Stieltjes measures on $(a, b]$, that is, μ_F is the only Borel measure on $(a, b]$ such that $\mu_F(a, x] = F(x) - F(a)$, $a < x \leq b$, and μ_G is defined analogously. Prove the integration by parts formula

$$\int_{(a,b]} F(x)d\mu_G(x) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G(x)d\mu_F(x).$$

Hint: Reduction to the case of μ_F and μ_G positive (and μ_F atomless), which does require using an important theorem, will allow you to apply a theorem on product measures. Check carefully and explicitly the hypotheses of any theorems you invoke.