Applied Math Prelim, August 2014

(1) Let H be a Hilbert space and $K : H \to H$ is linear, bounded and compact. Show that the range of $(I + K)$ is closed.
(it is a part of the proof that injective of $I + K$ implies surjective.)

(2a) Find the Green's function $G(x, y)$ for the operator A where

$$Au \equiv u'' - u$$

with $u'(0) = u(1) = 0$.
(2b) Define $T : L^2(0, 1) \to L^2(0, 1)$ such that for any $f \in L^2(0, 1)$,

$$(Tf)(x) = \int_0^1 G(x, y) f(y)\, dy\ .$$

Explain what spectral theorem is and why it is applicable.
(2c) Show that $\|T\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of T}\}$.
(2d) Compute $\|T\|$. (hint: find eigenvalues of A).

(3a) Let X and Y be Hilbert spaces and $F : X \to Y$. Give the definition for F being Frechet differentiable at $u_0 \in X$.

(3b) Let $F : L^4(0, 1) \to L^2(0, 1)$ such that for any $u \in L^4(0, 1)$, and any $x \in [0, 1]$, we have

$$(F(u))(x) \equiv (u(x))^2\ .$$

Show that F is well defined and is Frechet differentiable at any $u \in L^4(0, 1)$. Find such a derivative.

(4) Let $u : \mathbf{R}^3 \setminus \{0\} \to \mathbf{R}$ be defined by

$$u(x) \equiv -\frac{1}{4\pi|x|}\ .$$

Compute $\Delta u \equiv (\partial_{xx} + \partial_{yy} + \partial_{zz})u$ in distribution sense.

(5a) What is the definition of weak convergence of a sequence $\{x_n\}$ in a Hilbert space H;

(5b) Prove that the weak limit of $\{x_n\}$ is unique;

(5c) Let $T : H \to H$ be linear bounded compact operator and $x_n \rightharpoonup x_0$ weakly. Show that $Tx_n \to Tx_0$. (hint: a weakly convergence sequence is bounded).