Complex Functions Prelim, August 2014.

Below $\mathbb{D}$ denotes the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
In all cases the word "analytic" is used interchangeably with "holomorphic".

1. Suppose $h(x, y)$ is harmonic on $\mathbb{D}$ and has no zeros. Let $g(x, y)$ be its harmonic conjugate. Show that
$$f(x, y) = \log\Big(\big(h(x, y)\big)^2 + \big(g(x, y)\big)^2\Big)$$
is harmonic on $\mathbb{D}$. *(Hint: you are not intended to compute lots of derivatives.)*

2. Find the number of zeros of the function $f(z) = 2z^5 + 8z - 1$ in the annulus $1 < |z| < 2$.

3. Suppose $f(z) = \sum_0^\infty a_n z^n$ is analytic in $\{z : |z| < R\}$ and continuous in $\{z : |z| \leq R\}$, and let $M = \max_{|z| \leq R} |f(z)|$. Show that $|a_n| R^n \leq M$ for all n, and, more generally, $\sum_0^\infty |a_n|^2 R^{2n} \leq M^2$. *(Hint: For the latter consider $\int_{|z|=R} |f(z)|^2 \frac{dz}{z}$.)*

4. Let $\mathcal{F}$ be a family of holomorphic functions on $\mathbb{D}$ so that for any $f \in \mathcal{F}$,
$$|f'(z)| \left(1 - |z|^2\right) + |f(0)| \leq 1,$$
for all $z \in D$. Prove that $\mathcal{F}$ is a normal family on $\mathbb{D}$.

5. Suppose that f and g are analytic on $\mathbb{D}$ and satisfy
$$f(z) = z^3 g(z) - z.$$
   (a) If you know $f(\mathbb{D}) \subset \mathbb{D}$ show that $g(z) \equiv 0$.
   (b) If instead you know $f(\mathbb{D}) \supset \mathbb{D}$ and f is injective on $\mathbb{D}$, show that this also implies $g(z) \equiv 0$. *(Hint: It may help to consider the inverse of f.)*

6. Suppose f is an entire function that satisfies an equation of the form
$$\sum_{j=1}^{n} p_j(z) \big(f(z)\big)^j = 0$$
where the coefficients $p_j(z)$ are non-zero polynomials. Show that f is a polynomial.

   Hints: There is an elegant proof using the Casorati-Weierstrass theorem. Alternatively, it may help to prove one or both of the following:

   (a) If f is entire but it not a polynomial then for any $k \in \mathbb{N}$ there is a sequence $z_l \to \infty$ for which $z_l^{-k} f(z_l) \to \infty$.
   (b) There is a $k \in \mathbb{N}$ and a number M such that if $|z| \geq M$ and $|w| \geq 1$ then $\sum_j p_j(z) z^{kj} w^j$ cannot be zero.