Abstract Algebra Prelim Aug. 2015

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let R be a commutative ring (with identity) and I and J be ideals in R such that $I + J = R$.

   (a) Prove $R/(I \cap J) \cong R/I \times R/J$ as commutative rings ("Chinese remainder theorem").

   (b) Prove $IJ = I \cap J$, where IJ is defined to be the ideal in R generated by the set of products xy where $x \in I$ and $y \in J$.

   Be sure to use the condition $I + J = R$ in both (a) and (b); neither part is true in general without that.

2. Let G be a finite group and p be an odd prime number such that

   - every nontrivial element of G has order either 2 or p,
   - there are elements of both orders 2 and p,
   - there is a unique subgroup of order p.

   Letting $a \in G$ have order 2 and $b \in G$ have order p, prove $aba^{-1} = b^{-1}$ and every element of G is $b^i a^j$ where $0 \leq i \leq p-1$ and $0 \leq j \leq 1$. (Thus G has order $2p$. Do not assume that.)

3. Let G be a group whose subgroups are totally ordered by inclusion: for all subgroups H and K, either $H \subset K$ or $K \subset H$.

   (a) Prove every element of G has finite order.

   (b) Prove G is abelian. (This part does not depend on the previous one.)

   (c) If G is not trivial, prove the order of every element is a power of the same prime number.

4. Let F be a field. Using the fact that the polynomial ring $F[x]$ is a PID, prove that the Laurent polynomial ring $F[x, 1/x] = F[x][1/x] = \{\sum_{n=a}^{b} c_n x^n : a \leq b \text{ in } \mathbf{Z}, c_n \in F\}$ is a PID.

5. (a) State the classification of finitely generated abelian groups.

   (b) In $\mathbf{Z}^3$ let $H = \mathbf{Z}(2,2,6) + \mathbf{Z}(2,6,2)$. Describe the structure of the quotient group $\mathbf{Z}^3/H$ using the classification from part a.

6. Give examples as requested, with brief justification.

   (a) A finite group G that is generated by its subset of cubes $\{g^3 : g \in G\}$ but not all elements of G are cubes.

   (b) A commutative ring R (with identity) and ideals I and J in R such that $I + J \neq R$ and $IJ \neq I \cap J$.

   (c) A 2×2 real matrix without any real eigenvectors.

   (d) The statement of a theorem whose proof uses Zorn's lemma.