Topology Prelim, August 2015

Problem 1. Let X be a topological space and let A be a subset of X. Either prove the following statement, or give a counter-example.

1. If A is connected, then the closure $\overline{A}$ is connected.

2. If A is connected, then its interior $\mathrm{Int}(A)$ is connected.

Problem 2. Define the equivalence relation on $\mathbb{R}$ such that $x \sim y$ if $x - y$ is rational. Let $\mathbb{R}/\sim$ be the quotient space with the quotient topology. Show that $\mathbb{R}/\sim$ is not Hausdorff.

Problem 3. Let X, Y be topological spaces. Assume that Y is Hausdorff. Let $f, g : X \to Y$ be continuous functions. Suppose that there exists a dense subset D of X such that $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$.

Problem 4. A topological space X is said to be *contractible* if the identity map $\mathrm{Id}_X : X \to X$ is null-homotopic, i.e. homotopic to a constant map.

1. Show that any convex subset of $\mathbb{R}^n$ is contractible.

2. Let Y be a topological space. Show that if X is contractible, then any map $f : X \to Y$ is null-homotopic.

Problem 5. Let E, X be topological spaces. Assume that E is connected. Let $q : E \to X$ be a covering map with $q^{-1}(x)$ finite and non-empty for all $x \in X$. Show that E is compact if and only if X is compact.

Problem 6. Let $n \geq 3$. Suppose that M is a connected n-dimensional manifold, and $p \in M$. Show that the inclusion $M \setminus \{p\} \hookrightarrow M$ induces an isomorphism between their fundamental groups $\pi_1(M \setminus \{p\}) \cong \pi_1(M)$.