Abstract Algebra Prelim Jan. 2008

1. Let G be a group of order 21.

   a) Use the Sylow theorems to show G has a normal subgroup of order 7 and to determine how many subgroups it could have of order 3.

   b) If G has a normal subgroup of order 3, show G is abelian.

   c) Use semidirect products to construct an example of a nonabelian group of order 21.

2. Let I and J be ideals in a commutative ring R such that $I + J = (1)$.

   a) Define multiplication of ideals and show $IJ = I \cap J$.

   b) Show $R/(I \cap J) \cong R/I \times R/J$ as commutative rings. (Part a is not needed.)

3. a) Show $\mathbf{Z}[i]$ is a PID.

   b) Show any nonzero prime ideal in a PID is a maximal ideal.

4. Let H and K be subgroups of G, and let G act on the set $G/H \times G/K$ in a diagonal manner: $g(aH, bK) = (gaH, gbK)$. (We don't assume H or K is normal, so G/H and G/K are just sets, not necessarily groups.)

   a) Show the stabilizer subgroup of $(aH, bK) \in G/H \times G/K$ is $aHa^{-1} \cap bKb^{-1}$.

   b) If H and K have *finite* index in G, use the action of G on $G/H \times G/K$ to show there is a normal subgroup $N \lhd G$ of finite index contained in $H \cap K$.

5. State Zorn's lemma, then state a theorem whose proof uses Zorn's lemma, and then give the proof of that theorem. (Be sure to verify the conditions of Zorn's lemma are satisfied within the proof).

6. Give examples of

   a) a character of order 4 on the additive group $\mathbf{Z}/12\mathbf{Z}$ (either give a formula or a table of values). Recall a character of a finite abelian group is a homomorphism from the group to $\mathbf{C}^\times$.

   b) a solvable group G where $G' \neq \{e\}$ and $G'' = \{e\}$, with a computation of G'. (The notation G' is the commutator subgroup of G, and $G'' = (G')'$.)

   c) a maximal ideal in $\mathbf{Z}[X]$ which contains the ideal $(X^2 + 1)$.

   d) a unit other than ± 1 in $\mathbf{Z}[\sqrt{10}]$, along with its inverse.