Real Analysis PhD Exam, January 2008

Do 4 out the following 5 questions.

1. Find the limit as $n \to \infty$ or say that the limit does not exist, and justify your answer.

(a)

$$\int_0^\infty ne^{-nx} \sin(1/x)\, dx.$$

(b)

$$\int_0^1 \frac{1+n^2x^2}{(1+x)^n}\, dx.$$

2. Let μ be a finite measure on $\mathbb{R}$ and define $F(x) = \mu((-\infty, x])$. Show

$$\int [F(x+c) - F(x)]\, dx = c\mu(\mathbb{R}).$$

3. Suppose f is a positive integrable function on $[0,1]$ and define a measure μ on the Borel σ-algebra by

$$\mu(A) = \int_A f(x)\, dx.$$

Prove without using the Radon-Nikodym theorem that

$$\int_B \frac{1}{f(x)}\, \mu(dx) = m(B)$$

for every Borel set B, where $m(B)$ is the Lebesgue measure of B.

4. Let $1 \le p < \infty$. Prove that if a sequence of functions $f_n \in L_p(X, \mathcal{X}, \mu)$, $n \in \mathbb{N}$, is Cauchy in the L_p-norm, then there exists $f \in L_p$ such that $f_n \to f$ in the L_p-norm (that is, L_p is complete).

5. Say that $F : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz with constant M if

$$\sup_{x \neq y} |F(x) - F(y)|/|x-y| \le M.$$

Prove that a measurable function F is Lipschitz with constant M if and only if both F is absolutely continuous and $|F'| \le M$ a.s.