Some complex analysis prelim questions

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1. Suppose f is a nonconstant entire function such that $f \circ f(z) = f(z)$ for all z. Prove that f must be the identity function.

2. Suppose f is entire, $f(0) = 0$ and

$$|f(z)| \leq e^{1/|z|}$$

for all $z \neq 0$. Prove that f is identically 0.

3. Suppose for each n that f_n is a bounded continuous real-valued function on the unit circle $\{z : |z| = 1\}$. Suppose for each n that u_n is a function that is continuous on the closed unit disk $\{z : |z| \leq 1\}$, is harmonic in the open unit disk $\{z : |z| < 1\}$, and agrees with f_n on the unit circle. Show that $\{f_n\}$ is an equicontinuous family on the unit circle if and only if $\{u_n\}$ is an equicontinuous family on the closed unit disk.

4. Use residues to evaluate the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx.$$

5. Let $D = \{z = x + iy : 0 < y < 1, x > 0\}$. Find a conformal mapping of D onto the open unit disk.

6. Suppose that for each n the function f_n is analytic in the open unit disk, $|f_n(0)| \leq 1$, and for each $r < 1$ satisfies

$$\int_{|z|=r} |f_n(z)|^2 \; |dz| \leq 1.$$

Show that every subsequence of $\{f_n\}$ has a further subsequence which converges to a finite analytic function uniformly on each compact subset of the open unit disk.

7. Suppose for each n the function f_n is analytic on the open unit disk D and has exactly one zero in D. Suppose the sequence $\{f_n\}$ converges to f uniformly on each compact subset of the unit disk.

(a) Show that either f is identically zero on D or else has at most one zero in D.

(b) Give an example of a sequence $\{f_n\}$ where the limit function has no zeros in D.