MATH 5510 – Preliminary Examination.

January 18, 2010

Instructions: Answer two out of the three questions. **You do not have to prove results which you rely upon, just state them clearly !**

Q1) (a) Suppose that $A^{(1)} = A$ is an invertible $n \times n$ matrix and that the Gaussian elimination algorithm with partial pivoting applied to $A^{(1)}$ produces the upper triangular matrix $A^{(n)}$. As usual, let $A^{(k)}$ be the renamed $A^{(k)}$ following any necessary row intrechanges before the k–th major step of the elimination so that

$$a_{i,j}^{(k+1)} = \begin{cases} a_{i,j}^{(k)}, & \text{when } i \le k,\ 1 \le j \le n, \\ 0 & \text{when } i \ge k+1,\ 1 \le j \le k, \\ a_{i,j}^{(k)} - a_{i,k}^{(k)} a_{k,j}^{(k)} / a_{k,k}^{(k)}, & \text{when } i, j \ge k+1. \end{cases}$$

Show that the total number of multiplication and division operations needed to reduce $A^{(1)}$ to $A^{(n)}$ is $(n^3 - n)/3$. [Hint: Recall that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.]

b) Suppose that all the leading principal minors of A are positive. Show that A has an LU–factorization with unit diagonal entries in L and positive diagonal entries in U.

c) Suppose now that no partial pivoting is necessary and that $A^{(1)} = \left(a_{i,j}^{(1)}\right)$ is **tridiagonal**, that is, $a_{i,j}^{(1)} = 0$ when $|i - j| > 1,\ 1 \le i, j \le n$. Show that each of $A^{(1)}, \ldots, A^{(n)}$ is tridiagonal.

d) Suppose that A is an $n \times n$ invertible matrix which admits an LU–factorization without pivoting. Partition A into:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

with $A_{1,1}$ being a $(k-1) \times (k-1)$ matrix. Knowing that $A_{1,1}$ is invertible (why?), show that the current active array which is the $(n-k+1) \times (n-k-1)$ matrix $A_k = \left(a_{i,j}^{(k)}\right)$, $i, j = k, \ldots, n$ is given by:

$$A_k = A_{2,2} - A_{2,1} A_{2,2}^{-1} A_{1,2}.$$

Assume now that in addition to A being invertible and admitting an LU–factorization without pivoting, A is Hermitian. Use this formula to deduce that A_k is also Hermitian, for $k = 1, \ldots, n$.

Q2) (a) Prove the de la Vallée–Poussin lemma: *Suppose that f is a real function on the interval $[a, b]$. Let Ω_n be the set of all polynomials of degree at most n on $[a, b]$ and let $P \in \Omega_n$. Suppose there exist $n+2$ partition points*

$$a \leq x_1 < x_2 < \ldots < x_{n+2} \leq b \qquad (*)$$

and $n+2$ positive numbers $\lambda_1, \ldots, \lambda_{n+2}$ such that

$$f(x_i) - P(x_i) = (-1)^{i+m} \lambda_i, \quad i = 1, \ldots, n+2,$$

for some fixed integer m, $m = 0$ or $m = 1$. Then

$$\min_{Q \in \Omega_n} \|f - Q\|_\infty \geq \min\{\lambda_1, \ldots, \lambda_{n+2}\}.$$

Explain in your own words why the de la Vallée–Poussin lemma implies that if there exist $n+2$ points $x_1, \ldots, x_{n+2}$ in the interval $[a, b]$ satisfying $(*)$ such that at these points the function f and the polynomial P satisfy that

$$f(x_i) - P(x_i) = (-1)^{i+m} \|f - P\|_\infty, \quad i = 1, \ldots, n+2,$$

for some fixed integer m, $m = 0$ or $m = 1$, then P is the best approximation polynomial for f in Ω_n.

(b) If the function $f(x) = e^x$ is approximated on the interva; $[-1, 1]$ by a 4–th order Maclaurin polynomial, the resulting approximation is given by:

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

and the error is given by:

$$R_4(x) = \frac{x^5 f^{(5)}(\xi_x)}{5!},$$

for some point $\xi_x \in [-1, 1]$. Show that if $P_4(x)$ is now best approximated by a polynomial, call it $Q_3(x)$, of order at most 3 on the interval $[-1, 1]$, then

$$|f(x) - Q_3(x)| \leq |f(x) - R_4(x)| + |R_4(x) - Q_3(x)| \leq 0.03.$$

(c) Let S be the natural cubic spline that interpolates $f \in C^2[a, b]$ at the knots:

$$a = x_0 < x_1 < \ldots < x_N = b.$$

Show that

$$\int_a^b (S''(x))^2 dx \leq \int_a^b (f''(x))^2 dx,$$

and give an interpretation of this result.

3) (a) Let $\sum_{k=1}^{n} \alpha_k f_k$ be a Guassian quadrature scheme based on a system of orthogonal polynomials $\{p_j\}_{j=1}^{n}$ (with respect to the weight function $w(x) = 1$) on the interval $[a, b]$. Show that the weights $\alpha_1, \ldots, \alpha_n$ are positive.

(b) The Simpson's **Three–Eighths** quadrature rule for approximating $\int_a^b f(x)dx$ is a closed Newton–Cotes rule obtained by diving the interval $[a, b]$ into 3 subintervals of equal length h and approximation f on $[a, b]$ by an interpolation polynomial on the 4 points x_0, x_1, x_2, and x_3, where $x_0 = a$, $x_3 = b$, and $h = x_i - x_{i-1}$, for $i = 1, 2, 3$. Show that Simpson's Three–Eighths rule is give by:

$$I_4(f) = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3].$$

For approximating $\int_0^\pi \sin(x)dx = 2$, compare the usual Simpson's rule with his Three–Eighths rule.

Without carrying out a detailed error analysis for Simpson's Three–Eighths rule and comparing it with the error analysis for his usual rule, in what sense can we regard the the Three–Eighths rule to be superior to the usual one. Justify your answer.