Complex Functions Prelim, January 2015.

- Below $\mathbb{D}$ denotes the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- In all cases the word "analytic" is used interchangeably with "holomorphic".
- If you use a known result in your working **you must clearly state the result and make it apparent that you have verified all hypotheses needed for that result**.

1. Let
$$f(z) = \frac{1}{1+z^2} - \frac{z}{2-z}.$$
Write all of the Laurent series representations in z for the function f. For each representation, clearly state the region on which it is valid.

2. (a) State and prove the maximum principle for holomorphic functions.
   (b) Suppose f and g are *non-vanishing* holomorphic functions on $\mathbb{D}$ which extend continuously to the closed unit disc. If $|f(z)| = |g(z)|$ on the boundary $\{z : |z| = 1\}$, show that $|f(z)| = |g(z)|$ on the whole disc. Hence show there is λ with $|\lambda| = 1$ such that $f(z) = \lambda g(z)$ for all z.

3. (a) Show that $|z^3 - z + 1| > |z|$ when z lies on the imaginary axis in $\mathbb{C}$.
   (b) Determine the number of roots of $z^3 - z + 1 = ze^z$ that lie in the left half-plane in $\mathbb{C}$ (i.e. the set $z = x + iy$ with $x < 0$).

4. Let $\Omega \subsetneq \mathbb{C}$ be an open, simply-connected set that is not all of $\mathbb{C}$. Fix two points $w, z \in \Omega$ and let
$$C(\Omega, w, z) = \sup\Big\{|f'(w)| \text{ such that } f : \Omega \to \Omega \text{ is analytic and } f(w) = z\Big\}$$
   (a) In the special case $\Omega = \mathbb{D}$ and $w = z = 0$ the value $C(\mathbb{D}, 0, 0)$ is given by a famous result. State and prove this result and give the value $C(\mathbb{D}, 0, 0)$.
   (b) Compute $C(\mathbb{D}, 0, z)$ as a function of z.
   (c) Let $\Omega \subsetneq \mathbb{C}$ be an open and simply-connected set which is not all of $\mathbb{C}$, and suppose $w, z \in \Omega$. Prove that there is a real number $C < \infty$ so that if $f : \Omega \to \Omega$ is analytic and $f(w) = z$ then $|f'(w)| \le C$. (This shows $C(\Omega, w, z) < \infty$.)

5. If f is analytic and injective on a neighborhood of 0 and $f(0) = 0$, show that there is a single-valued branch of $\sqrt{f(z^2)}$ on a (possibly smaller) neighborhood of 0.