Topology Prelim, January 2015

You may use any result that has been proven in class, unless the question directly asks you to prove the result. Please, state the results you are using.

Problem 1. Let A and B denote subsets of a topological space X. Prove that
$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Problem 2. Let $f_1, f_2 : X \to Y$ be continuous maps from a topological space X to a Hausdorff space Y. Show that the set of points $\{x \in X : f_1(x) = f_2(x)\}$ is a closed set.

Problem 3. Let X be a topological space, and let $A \subseteq X$ be a subset. Denote by $\text{Int}(A)$ and ∂A the interior and boundary of A, respectively. Either prove the following statement, or give a counter-example.

1. If A is connected, then $\text{Int}(A)$ is connected.

2. If both $\text{Int}(A)$ and ∂A are connected, then A is connected.

Problem 4. Let $q : E \to X$ be a covering map with $q^{-1}(x)$ finite and nonempty for all $x \in X$. Show that E is compact if and only if X is compact.

Problem 5. Let $\mathbb{P}^2$ denote the (real) projective plane. Prove that any continuous map $f : \mathbb{P}^2 \to \mathbb{S}^1$ is null-homotopic, i.e. homotopic to a constant map.

Problem 6. Let $n \geq 3$ be an integer. Suppose M and N are connected n-dimensional manifolds. Prove that the fundamental group of the connected sum $M \# N$ is isomorphic to $\pi_1(M) * \pi_1(N)$.