COMPLEX ANALYSIS PRELIM, JANUARY 2016

Denote $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, and $A \setminus B = \{x \in A; x \notin B\}$ for two sets A and B.

1. Does there exist a function f, holomorphic in $\mathbb{C} \setminus \{0\}$, such that
$$|f(z)| \geq \frac{1}{|z|^{9/10}}$$
for all $z \in \mathbb{C} \setminus \{0\}$? Prove your assertion.

2. Let $\mathrm{Aut}(\mathbb{D})$ be the group of holomorphic automorphisms of $\mathbb{D}$ and let Id be the identity map.
   (i) For each $b \in \mathbb{D}$, construct a map $\varphi \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}\}$ such that b is a *fixed point* of φ, i.e., $\varphi(b) = b$.
   (ii) Does there exist a map $\psi \in \mathrm{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}\}$ such that ψ have two distinct fixed points in $\mathbb{D}$? Prove your assertion.

3. Let $G = \{z \in \mathbb{C}; |z| > 2\}$ and $f(z) = 1/(z^4 + 1)$. Is there a complex differentiable function on G whose derivative is $f(z)$? Prove your assertion.

4. Let f be a holomorphic function in $\mathbb{D}$. Suppose that $|f(z)| \leq 1/(1 - |z|)$ for all $z \in \mathbb{D}$. Prove that
$$|f'(z)| \leq \frac{4}{(1 - |z|)^2} \quad \text{for all } z \in \mathbb{D}.$$

5. How many zeros counting multiplicities does the polynomial
$$p(z) = z^5 + z^3 + 5z^2 + 2$$
have in the region $\{z \in \mathbb{C}; 1 < |z| < 2\}$? Prove your assertion.

6. Let d be the distance function on the Riemann sphere $\widehat{\mathbb{C}}$ given by
$$d(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \quad d(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}},$$
for all $z, w \in \mathbb{C}$.
   (i) Prove that
$$\frac{1}{2}\left|\frac{1}{z} - \frac{1}{w}\right| \leq d(z, w) \leq \left|\frac{1}{z} - \frac{1}{w}\right|$$
   for all $z, w \in \mathbb{C}$ and $|z| \geq 1, |w| \geq 1$.
   (ii) Let Ω be a domain in $\mathbb{C}$ and $\{f_k\}$ a sequence of holomorphic functions on Ω. Suppose that $\{f_k\}$ converges, with respect to d, uniformly on every compact subset of Ω to a function g which take values in $\widehat{\mathbb{C}}$. Show that either g is a holomorphic function on Ω, or $g \equiv \infty$.