Abstract Algebra Prelim Jan. 2017

Justify all your steps. You may use any results that you know unless the question says otherwise, but don't invoke a result that is essentially equivalent to what you are asked to prove or is a standard corollary of it.

1. Let G be a finite group and p be a prime number.
   (a) Define a p-Sylow subgroup of G and state the Sylow theorems for G.
   (b) If H is a p-Sylow subgroup of G and N is a normal subgroup of G, prove $H \cap N$ is a p-Sylow subgroup of N. (Hint: Consider the order of $H \cap N$ relative to that of H and N.)

2. (a) Let p be a prime. Prove the group $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ has order $(p^2-1)(p^2-p)$.
   (b) Construct a non-trivial semidirect product $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes_\varphi (\mathbb{Z}/3\mathbb{Z})$. That is, construct a semidirect product where $\varphi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2)$ is not trivial and explicitly describe the group law in the semidirect product. (Hint: $\mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2) \cong \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$.)
   (c) Show the only semidirect product $(\mathbb{Z}/7\mathbb{Z})^2 \rtimes_\varphi (\mathbb{Z}/5\mathbb{Z})$ is the trivial one.

3. Let $i = \sqrt{-1}$ in $\mathbb{C}$.
   (a) Show that $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-2}]$ are isomorphic as *additive groups*.
   (b) Show that $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-2}]$ are not isomorphic as *rings*.

4. (a) For an integral domain A, define an *irreducible element* of A, a *prime element* of A and what it means to say A is a *unique factorization domain* (UFD).
   (b) Prove that in a UFD every irreducible element is prime.

5. Let R be an integral domain. An element $s \in R$ that is not zero and not a unit is called "special" if, in the quotient ring $R/(s)$, each coset is represented by 0 or a unit from R: for each $a \in R$ we have $a \equiv 0 \bmod (s)$ or $a \equiv u \bmod (s)$ where $u \in R^\times$.
   (a) If $s \in R$ is special, prove that the principal ideal (s) in R is maximal.
   (b) In $\mathbb{Z}[i]$ prove $1+i$ is special and 3 is not special.
   (c) Prove that there are no special elements in $\mathbb{Z}[x]$. (Hint: Apply the definition of special with $a = 2$ and with $a = x$.)

6. Give examples as requested, with justification.
   (a) A finite group of even order that does not have a subgroup of index 2.
   (b) A generator of the character group of $\mathbb{Z}/4\mathbb{Z}$.
   (c) An irreducible polynomial of degree 3 in $(\mathbb{Z}/3\mathbb{Z})[x]$.
   (d) A prime factorization of 10 in $\mathbb{Z}[i]$.